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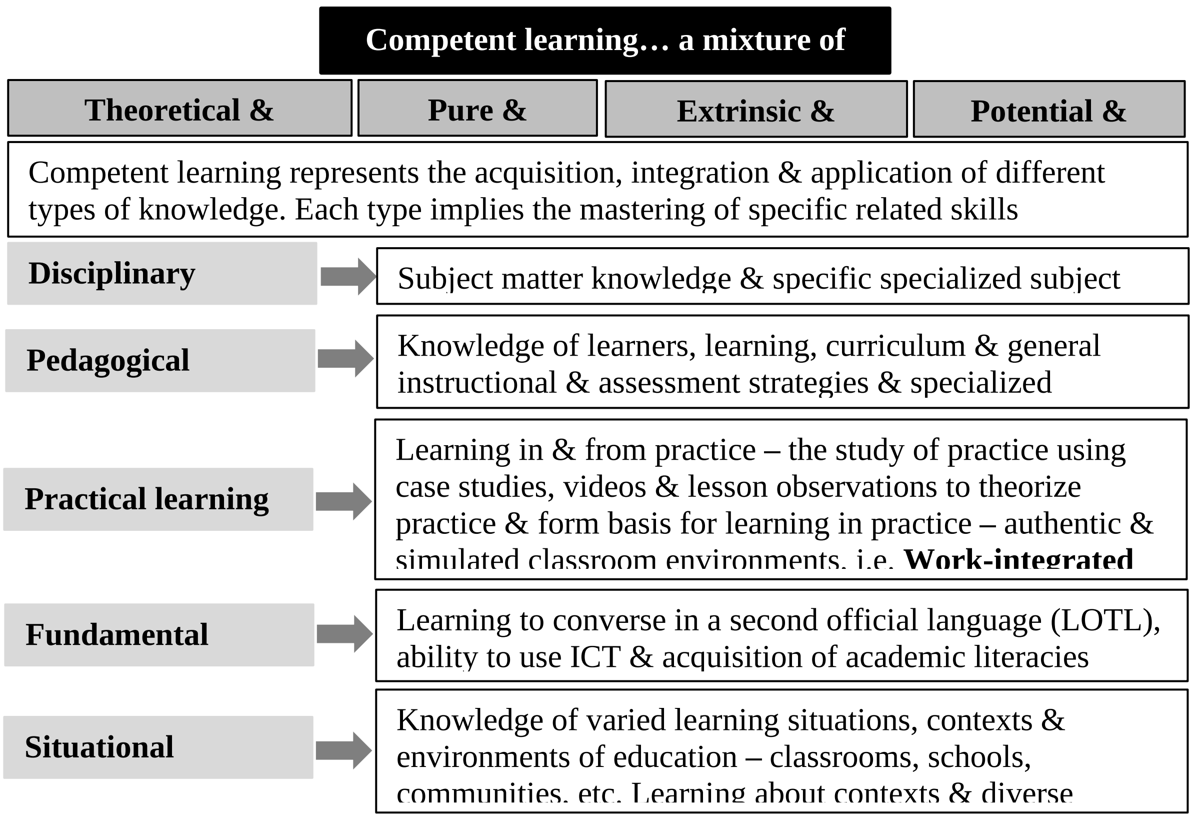
TEACHING & LEARNING DEVELOPMENT CAPACITY IMPROVEMENT PROGRAMME

Co-funded by the European Union

PRIMARY TEACHER EDUCATION (PrimTEd) PROJECT
GEOMETRY AND MEASUREMENT WORKING GROUP
DRAFT FRAMEWORK FOR A TEACHING UNIT

Preamble

The general aim of this teaching unit is to empower pre-service students by exposing them to geometry and measurement, and the relevant pedagogical content that would allow them to become skilful and competent mathematics teachers. The depth and scope of the content often go beyond what is required by prescribed school curricula for the Intermediate Phase learners, but should allow pre-service teachers to be well equipped, and approach the teaching of Geometry and Measurement with confidence. Pre-service teachers should essentially be prepared for Intermediate Phase teaching according to the requirements set out in MRTEQ (Minimum Requirements for Teacher Education Qualifications, 2019). “MRTEQ provides a basis for the construction of core curricula Initial Teacher Education (ITE) as well as for Continuing Professional Development (CPD) Programmes that accredited institutions must use in order to develop programmes leading to teacher education qualifications.” [p6].



Target Audience

For utilisation by teacher-educators for the education of Intermediate Phase mathematics pre-service teachers.

“Big Ideas” in Geometry and Measurement

“A Big Idea refers to core concepts, principles, theories, and processes that should serve as the focal point of curricula, instruction, and assessment. Big Ideas reflect expert understanding and anchor the discourse, inquiries, discoveries, and arguments in a field of study. They provide a basis for setting curriculum priorities to focus on the most meaningful content.” (McTighe & Wiggins, 2004).
This section will address properties, measurement and transformations as constituents of the concept of geometry and measurement big ideas.

Rationale

Transformation is an explicit and deliberate inclusion, seeking to develop a specific concept at the Intermediate Phase level as part of the process of developing the learners’ understanding of school mathematics. Transformation as a mathematical concept incorporates the development of spatial reasoning. Spatial reasoning cuts across both the Space and Shape and Measurement content areas of the school curriculum. This is to be kept in mind while mediating the content with learners so as to avoid a blinkered conception of the concept of transformation and its utility in understanding the field of mathematics. For this reason, the concept of transformation is broken up into the following sub-topics in this exposition of the content.

The South African school curriculum (CAPS) determines the content relating to transformations in the Intermediate Phase. What follows are extracts from the curriculum document that determines the scope of what learners in the Intermediate Phase need to learn relating to this topic.

3. SPACE AND SHAPE (GEOMETRY)

- The main progression in Space and Shape is achieved by a focus on new properties and features of 2D shapes and 3D objects in each grade.
- Giving learners opportunities to identify and describe features of 2D shapes and 3D objects, develops their abilities to be able to classify shapes and objects in the Senior Phase

TOPICS	GRADE 4	GRADE 5	GRADE 6
3.4 Transformations	**Build composite shapes** • Put 2-D shapes together to make different composite 2-D shape including some shapes with line symmetry.	**Use transformations to make composite shapes** • Make composite 2-D shapes including shapes with line of symmetry by tracing and moving a 2-D shape in one or more of the following ways: o rotation o translation o reflection	
	Tessellations • Pack out 2D shapes to make tessellating patterns including some patterns with line symmetry. **Describe Patterns** • Refer to lines, 2-D shapes, 3-D objects and lines of symmetry when describing patterns o in nature o from modern everyday life o our cultural heritage	**Use transformations to make tessellations** • Make tessellating patterns including some patterns with line symmetry by tracing and moving 2D in one or more of the following ways: o rotation o translation o reflection **Describe patterns** • Refer to lines, 2-D shapes, 3-D objects, lines of symmetry, rotations, reflections and translations when describing patterns: o in nature, o from modern everyday life o from our cultural heritage	**Enlargement and reductions** • Draw enlargement and reductions of 2D shapes to compare size and shape of: o triangles o quadrilaterals **Describe patterns** • Refer to lines, 2-D shapes, 3-D objects, lines of symmetry, rotations, reflections and translations when describing patterns o in nature, o from modern everyday life o from our cultural heritage

Curriculum and Assessment Policy Statement (CAPS) Intermediate Phase
Grades 4-6 Mathematics (2012)

The curriculum statement further defines the Intermediate Phase content focus for the Space and Shape content area with reference to ‘transformations’.

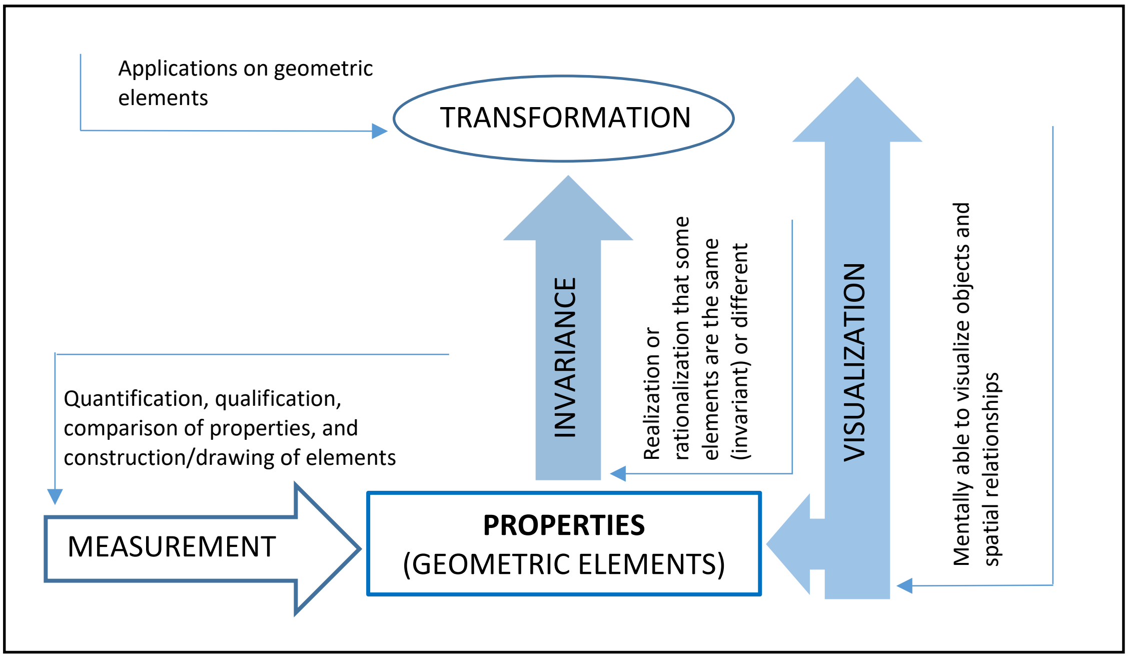
MATHEMATICS CONTENT KNOWLEDGE

Content Area	General content focus	Intermediate Phase specific content focus
3. Space and Shape (Geometry)	The study of Space and Shape improves understanding and appreciation of the pattern, precision, achievement and beauty in natural and cultural forms. It focuses on the properties, relationships, orientations, positions and transformations of two-dimensional shapes and three-dimensional objects.	• The learner's experience of space and shape in this phase moves from recognition and simple description to classification and more detailed description of features and properties of two-dimensional shapes and three dimensional objects. • Learners should be given opportunities to: o draw two-dimensional shapes and make models of three-dimensional objects; and o describe location, transformations and symmetry.

(CAPS, 2012)

From this, it becomes clear that transformation is an explicit and deliberate inclusion, seeking to develop a specific concept at the Intermediate Phase level as part of the process of developing the learners' understanding of school mathematics. Transformation as a mathematical concept incorporates the development of spatial reasoning. Spatial reasoning cuts across both the Space and Shape and Measurement content areas of the school curriculum. This is to be kept in mind while mediating the content with learners so as to avoid a blinkered conception of the concept of transformation and its utility in understanding the field of mathematics. For this reason, the concept of transformation is broken up into the following sub-topics in this exposition of the content.

Transformation – conceptual structure



The content in the national curriculum statement (CAPS) is structured in such a way that, in developing a conception of transformation, learners are exposed to the same aspects of the concept in each of the individual grades constituting the Intermediate Phase. The conceptual progression in each grade, along with an awareness of the conceptual structure and internal integration with related mathematics topics, should therefore be structured in such a way so as to facilitate a deep understanding of the concept transformation.

Topic	Grades	Sub-topics	Concepts and skills
Transformation	Grade 4	4.2 Transformations (1)	Construct and describe composite geometric elements.
		4.4 Transformations (2)	Identifying line symmetry and tessellating geometric elements.
	Grade 5	5.2 Transformations (1)	Constructing composite geometric elements, tessellations and patterns through translation, rotation and reflection.
		5.4 Transformations (2)	Employing translation, refection and rotation to tessellate, construct composite geometric elements and patterns.
	Grade 6	6.2 Transformations (1)	Describe how geometric elements can be transformed (translation, reflection and rotation) to develop tessellations.
		6.3 Transformations (2)	Draw enlargements and reductions of 2D geometric elements.

Content Standards for Transformations

Standard 7: Knowledge of Transformations

Knowledge of transformations in Geometry develop the ability to manipulate, visualize, recognize, identify invariance and variance among geometric elements in a variety of orientations and from different perspectives.

Sub-Standards:

7.1 Understanding and representing translations, reflections, rotations, and dilations of objects in the plane

7.2 Drawing and constructing representations of tessellations of two-dimensional geometric shapes or three-dimensional objects using transformations and a variety of tools

7.3 Comparing geometric patterns (tessellations) that share common characteristics (e.g. form, line, angle, vertex arrangement, space)

7.4 Demonstrating how (elements and principles) can be used to solve specific spatial visual problems

7.5 Planning and producing works of art applying mathematical techniques, and processes with skill, confidence, and sensitivity

Theories, Teaching Approaches and Methodology

Teaching does not occur in a vacuum. In fact, all teaching is informed by a single or a combination of theories. Both inductive and deductive approaches will be modelled with this teaching unit. Teaching methodology will include investigation, and discussion, hypothesizing, and modelling.

Commognition

(Sfard, 2001; Siyepu & Ralarata, 2014)

An analytical framework of the communicational attitude to cognition, which could be perceived as including both cognitive and socio-cultural methods. These methodologies perceive learning as a practice of becoming a partaker in a certain distinct discourse (considered a distinctive type of communication, made discrete by its range of admissible actions and the way these actions are harmonized with reactions (Sfard, 2001:28).

Realistic Mathematics Education

(Freudenthal, 1991; Gravemeijer, 1994)

The authentic activity of doing mathematics should primarily consist of unifying or mathematising subject matter taken from representativeness. Learners should therefore learn mathematics by mathematising subject matter from genuine contexts and their own mathematical activity rather than from the customary view of presenting mathematics to them as an off-the-rack system which are commonly applied.

Constructivism

(Williams & Matthews, 2003)

We construct knowledge from our own insights and experiences, which are dependently and independently mediated through our previous knowledge. Learning is the procedure by which human beings adjust to their realistic world.

Enactivism

(Varela, 1996)

The approach to teaching and learning primarily engages with the correlation between the learners and their familiar environment. It entails “knowing in action” (Reid, 2005:1). It is essentially about active engagement of learners in the learning of mathematics as a multifaceted process and endeavour.

Relational and Instrumental learning

(Skemp, R.R. (1976)

Relational understanding entails knowing what strategy works and why it works. It permits us to relate the strategy to the problem and possibly adjust the strategy to different problems. Instrumental understanding on the other hand demands memorization of facts or methods that work for which problems and which do not. It entails learning a different method for each new class of problems.

Van Hiele levels

(Van der Walle, J.A., et.al. 2008)

The van Hiele model is based on a five-level chain of command depicting habits of understanding spatial ideas. Thinking processes in geometric contexts are described by different levels, which explicitly describe our thinking processes regarding the categories of geometric concepts we think about, i.e. objects of thought.

Situated learning

(Adendorff, 2007; Elmholt, 2001)

This approach to teaching and learning promotes construction of knowledge by humans based on the progression of everyday experiences. Learning usually transpires as a purpose of a specific activity. It is situated due to the context in which it occurs as well as the culture implied.

Problem-solving

(Polya) (1887 – 1985)

Comprises the thought processes prolific in learning and doing mathematics. Sense-making and perseverence in solving problems are promoted in this approach to learning. Learners reason conceptually and quantitatively, construct feasible arguments, assess others’ reasoning, apply mathematical modelling, apply suitable tools purposefully, attend to accuracy, look for and apply structure and they investigate and verbalize consistency in repeated reasoning.

Exploration through guided-reinvention

The role of the teacher in applying guided-reinvention is to be a guide to the learner, but this guidance should facilitate discovery (Van Etten & Adendorff, 2001). Guided re-invention is one of the key principles of realistic mathematics education (Gravemeijer, 1994:90) according to which learners "should be given the opportunity to experience a process similar to the process by which mathematics was invented". The aim is not to begin "from abstract principles or rules with the aim to learn to apply these in concrete situations, nor does it focus on an instrumental type of knowledge" (Wubbels, Korthagen and Broekman, 1997:2). The main emphasis is rather on facilitating and enhancing knowledge construction. This is a more dynamic approach of dealing with mathematical concepts by deliberately guiding learners through a reinvention process (Freudenthal, 1991).

Relational rather than instrumental teaching

When teaching tessellations (stemming from transformations) there are two types of understanding that the teacher could develop in learners, namely relational and instrumental understanding. Relational understanding is considered more advantageous to learners as opposed to instrumental understanding (Skemp, 1978). Teaching for instrumental understanding implies emphasis on the rules and procedures to use and the application of skills to use these. Teaching for relational understanding on the other hand is getting the learner to know how to apply and use the rule, and having insight in why it works.

Suggested sequence of conceptual development activities

Unit	Topic	Focus
1	Understand & represent transformations	translations, reflections, rotations, and dilations of shapes in a plane & objects in space
2	Drawing and constructing representations	two-dimensional geometric shapes or three-dimensional objects using transformations and a variety of tools
3	Compare, discuss & describe geometric patterns	that share common characteristics (e.g. form, line, angle, vertex arrangement, space)
4	Demonstrating how (elements and principles)	can be used to solve specific spatial visual problems
5	Planning and producing works of art	works of art applying mathematical techniques, and processes with skill, confidence, and sensitivity

Conceptual Development Activities

Unit 1	Title of unit: Introduction of Transformations

Content standard:

7.1. Understanding and representing translations, reflections, rotations, and dilations of objects in the plane

Intent of this activity

If a shape is transformed, its appearance is changed. This can be done in a number of ways, including translation (slides), rotation (turns), reflection (flips) or a combination of these and enlargement and reduction. Transformations is an aspect of the mathematics school curriculum constituting the 'Space and Shape' content area and can be defined as follows:

'A process by which one figure, expression or function is converted into another of similar value.' (Apple Dictionary - Version 2.2.1).

'Changes in position or size of a shape: movements that do not change the size or shape of the objects transformed are called 'rigid motions' - translation, rotation and reflection. (Van De Walle, Karp, Bay-Williams. Elementary and Middle School Mathematics - Teaching Developmentally, 2008).

Translation

A translation, also known as a slide, is the rigid motion in which all points of the plane are moved the same distance in the same direction. An arrow drawn from a point P to its image point P completely specifies the two pieces of information required to define a translation: The direction of the slide is the direction of the arrow, and the distance moved is the length of the arrow. The arrow is called the slide arrow or translation vector (Long, et al, 2015: p.588).

Rotation

A rotation, also called a turn , is another basic rigid motion. One point of the plane—called the turn centre or the centre of rotation —is held fixed, and the remaining points are turned about the centre of rotation through the same number of degrees – the turn angle or angle of rotation (Long et al, 2015: p.588).

Reflection

The third basic rigid motion is a reflection, which is also called a flip or a mirror reflection. A reflection is determined by a line in the plane called the line of reflection or the mirror line . Each point P of the plane is transformed to the point P on the opposite side of the mirror line m and at the same distance from *m* (Long, et al, 2015: p.590).

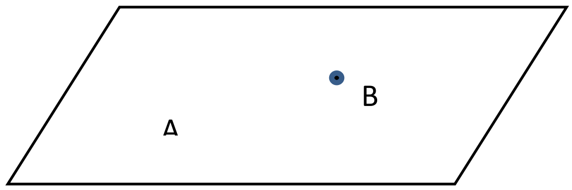
Glide–Reflections

The fourth, and last, basic rigid motion is the glide–reflection. As the name suggests, a glide–reflection combines both a slide and a reflection. The example most easily recalled is the motion that carries a left footprint into a right footprint, as depicted in Figure 11.7. It is required that the line of reflection, called the glide mirror, be parallel to the direction of the slide. The slide is usually called a glide, and its vector is called the glide arrow or glide vector. In Figure 11.7, the slide came before the reflection, but if the reflection had preceded the slide, the net outcome would have been the same. Note that the image of the left footprint under the glide–reflection is the blue right footprint. The green footprint is only an intermediate step used in completing the transformation (Long, et al, 2015: p.592).

Transformation: starting with point

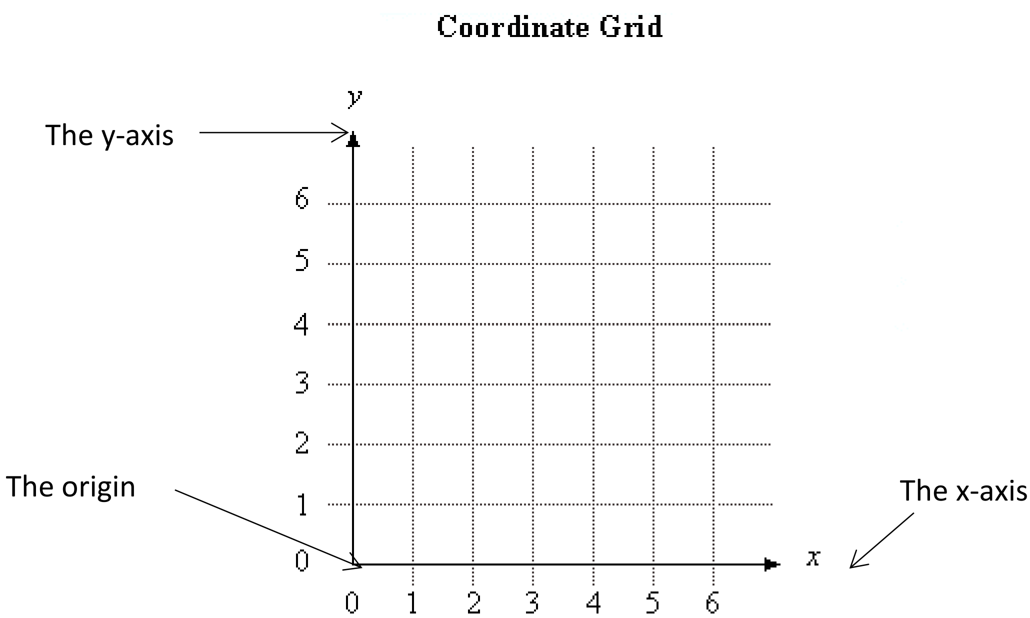
Before considering rigid transformation of 2D shapes or 3D objects, it makes sense to start with the most elementary concept in geometry, the point. The question that is asked intuitively is, What is meant by a point in geometry?

A **point** in **geometry** is an exact location on a 2D plane or a specific position in space (3D). It has no magnitude (size). This means it has no dimensions – no width, no length and no depth. A p**oint** is indicated by a dot and a capital letter. A line is **defined** as a line of **points** that extends infinitely in opposite directions. Points may be used to name a line, ray, line segment, angles, 2D shapes such as triangles or squares:



Plotting and finding a point on the Coordinate Grid

Plotting a point or coordinate graphing is a visual way for showing relationships between numerical values. The relationships are shown on a **coordinate grid**. A coordinate grid has two perpendicular lines, or **axes**, similar to number lines. The **horizontal axis** is called the ***x*-axis**. The **vertical axis** is called the ***y*-axis**. The point where the *x*-axis and *y*-axis intersect is called the **origin.**



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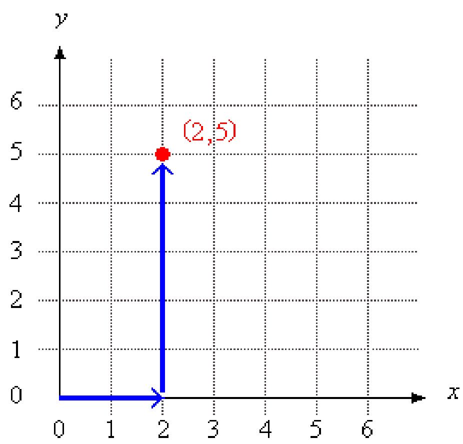
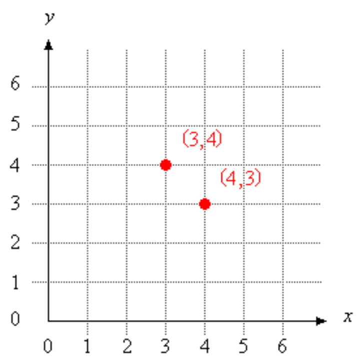
The numerical values on a coordinate grid are used to plot or locate points. Each point can be identified by an **ordered pair** of numbers; that is, a number on the *x*-axis called an ***x*-coordinate**, and a number on the *y*-axis called a ***y*-coordinate**. Ordered pairs are written in brackets (*x*-coordinate; *y*-coordinate). The origin is located at (0;0).

Observe the point on the grid below. What is its location or position? Its location (2;5) is shown on the coordinate grid. The *x*-coordinate is 2 (move from the origin, to spaces to the right). The *y*-coordinate is 5 (move from x = 2, five spaces upward).



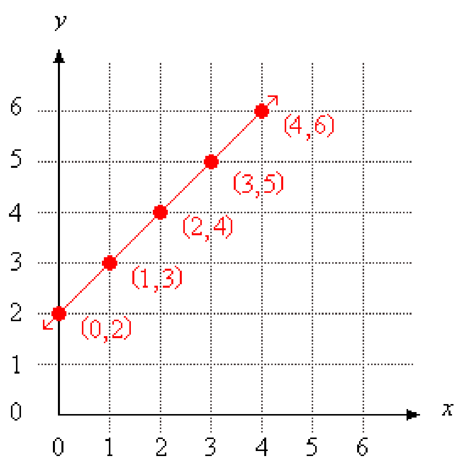
The order in which you write *x*- and *y*-coordinates in an ordered pair is important. The *x*-coordinate is written first, followed by the *y*-coordinate. By reversing the positions of the numbers in the coordinate pair you will change the position of the point. From the coordinate grid below, the ordered pairs (3,4) and (4,3) clearly refer to two different points!



The function table below indicates the *x*- and *y*-coordinates for five ordered pairs. The relationship between the *x*- and *y*-coordinates for each of these ordered pairs with this rule: the x-coordinate plus two equals the y-coordinate. You can also describe this relationship with the algebraic equation $x + 2 = y$.

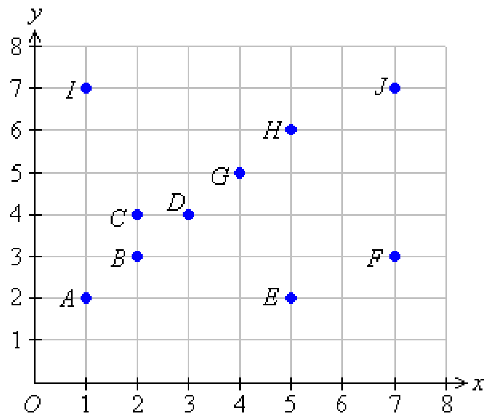
x-coordinate	x + 2 = y	y-coordinate	Ordered pair
0	0 + 2 = 2	2	(0;2)
1	1 + 2 = 3	3	(1;3)
2	2 + 2 = 4	4	(2;4)
3	3 + 2 = 5	5	(3;5)
4	4 + 2 = 6	6	(4;6)

To graph the equation x + 2 = y each ordered pair is located on a coordinate grid, then the points are connected. Notice that the graph forms a straight line. The arrows indicate that the line goes on in both directions. The graph for any simple addition, subtraction, multiplication, or division equation forms a straight line.



Example

Give the coordinates of each of the points shown on the Cartesian plane:

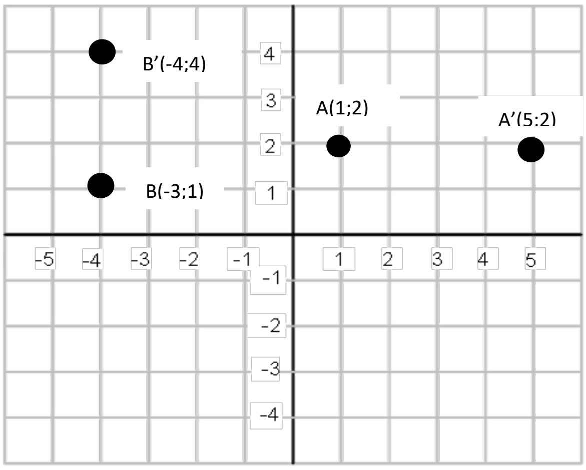


Solution:

A is 1 unit to the right of and 2 units above the origin. So, point *A* is (1;2).
B is 2 units to the right of and 3 units above the origin. So, point *B* is (2;3).
C is 2 units to the right of and 4 units above the origin. So, point *C* is (2;4).
D is 3 units to the right of and 4 units above the origin. So, point *D* is (3;4).
E is 5 units to the right of and 2 units above the origin. So, point *E* is (5;2).
F is 7 units to the right of and 3 units above the origin. So, point *F* is (7;3).
G is 4 units to the right of and 5 units above the origin. So, point *G* is (4;5).
H is 5 units to the right of and 6 units above the origin. So, point *H* is (5;6).
I is 1 unit to the right of and 7 units above the origin. So, point *I* is (1;7).
J is 7 units to the right of and 7 units above the origin. So, point *J* is (7;7).

Finding the distance between two points

Finding the distance between two or more points is important, for instance to determine how far the image of a point is situated from the pre-image. Determining the distance between points can be done in different ways. If two points are situated parallel to the x-axis, the smaller x-value can simply be subtracted from the greater x-value. The same applies if two points are situated parallel to the y-axis. In the diagram below the length of line segment $AA' \; is \; 6 - 1 \; = \; 4$ units and $BB' = 4 - 1 = 3$ units. Distance or length is always given as a positive value. Why?

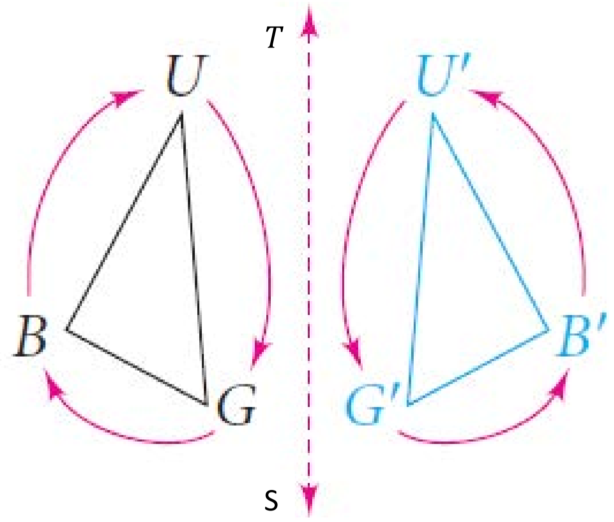


Basic transformations involving positive numerical values

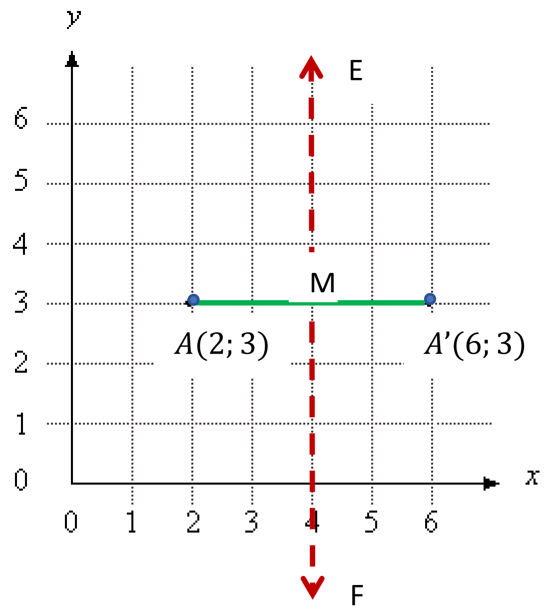
Some concepts related to transformation to remember are the following:

- **Transformation** of geometric shapes or objects is a change in its position, shape or size.
- **Pre-image** refers to the original shape or figure.
- The **image** is the shape or figure after having undergone transformation.
- **Rigid transformation** refers to transformation that does change or alter the size or shape of a geometric figure.
- **Similarity transformation** is a transformation type that changes the size but not the shape of geometric figures.
- An **isometry**: is a transformation that maintains congruency. This means that the transformation does not change the size of the figure or shape.

Reflection: A point can be reflected. Such a reflection occurs along a line or line segment called an axis or line of reflection. In this diagram point U(pre-image) is reflected along line ST to form U' (image).



In the sketch on the Cartesian plane below, point $A(2;3)$ is reflected on the line EF. Notice that point A is the pre-image and the reflected image A' are the same distance away from axis EF, meaning that $AM = A'M$.



Translation of a point on the grid: Point C(2;4), the pre-image is translated (shifted) resulting in the translated image C'(4;2) in the following way (x + 2;y-2).

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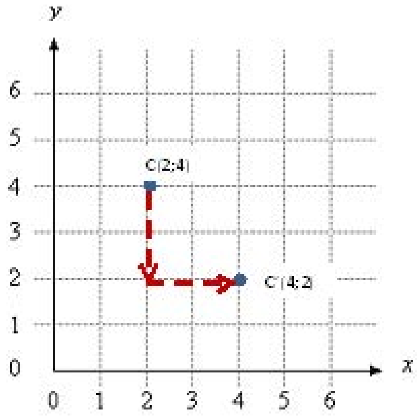
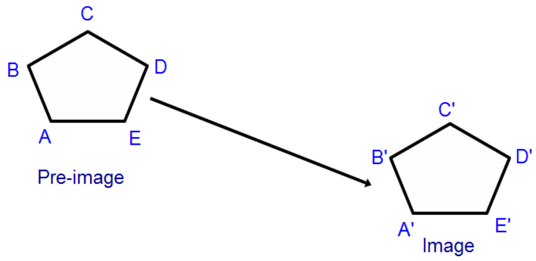
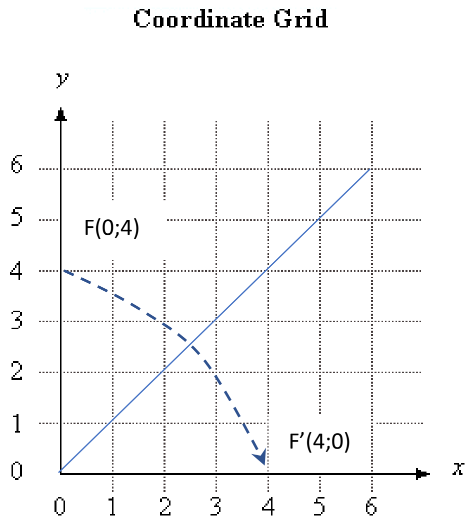
Translation thus is a transformation that maps all points of the pre-image the **<u>same distance</u>** in the **<u>same direction</u>** to form the image. The pre-image ***slides*** to a new location without "turning" or "flipping".

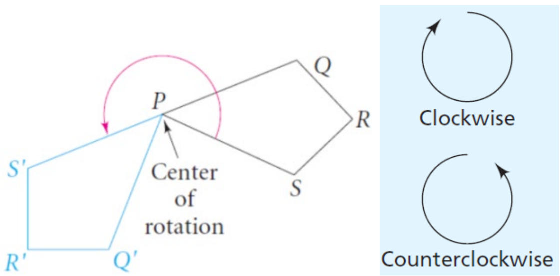




Rotation: A point can be rotated though an angle of a certain number of degrees, for example through 45º, 90º, 180º etc. about a given point The rotation can be clockwise or anti-clockwise. The point about which the shape or object is rotated can be inside or outside of the shape or object. The amount of rotation is called the angle of rotation and is measured in degrees. In the diagram below point F(0;4) is rotated through 90º clockwise with the rotated image being F'(4;0).



Rotation – a transformation where a figure ***"turns"*** around a point called the **centre of rotation**

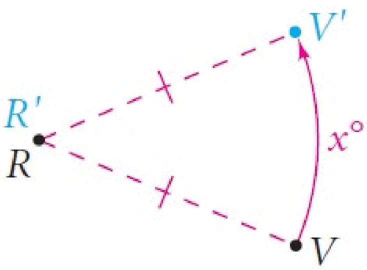


A ***rotation*** of $x°$ about a **center point R** is a transformation for which the following must be true:

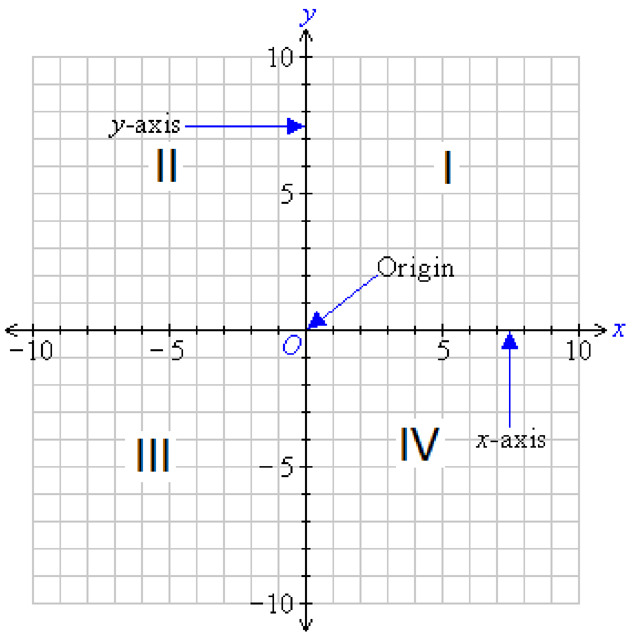
The image of R is itself.

$$m\angle VRV' = x°$$

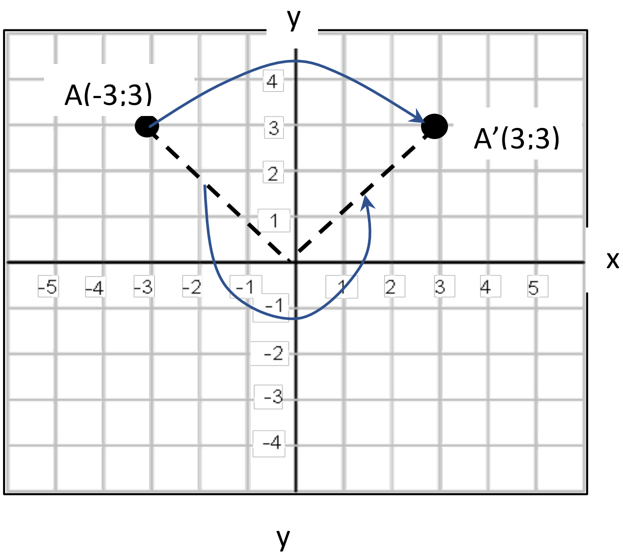
For any point V, $RV = RV'$



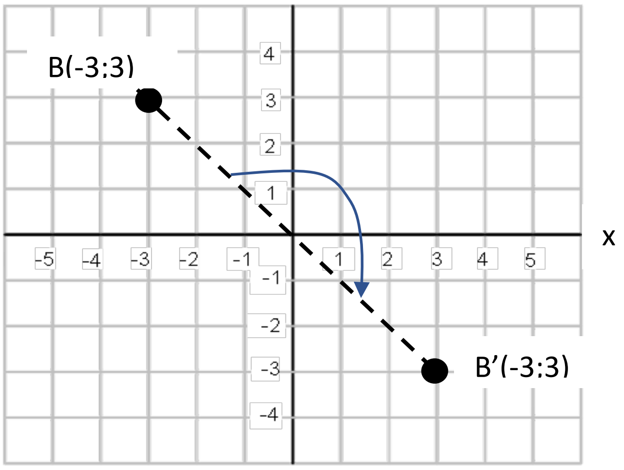
Thus far we have only worked with positive numerical values on part of the Cartesian plane. The sketch below shows the whole Cartesian plane consisting of four quadrants I to IV. The axes of a two-dimensional Cartesian system divide the plane into four infinite regions, called quadrants, each bounded by **two** half-axes. See the diagram of the Cartesian plane below



90º /270º rotation



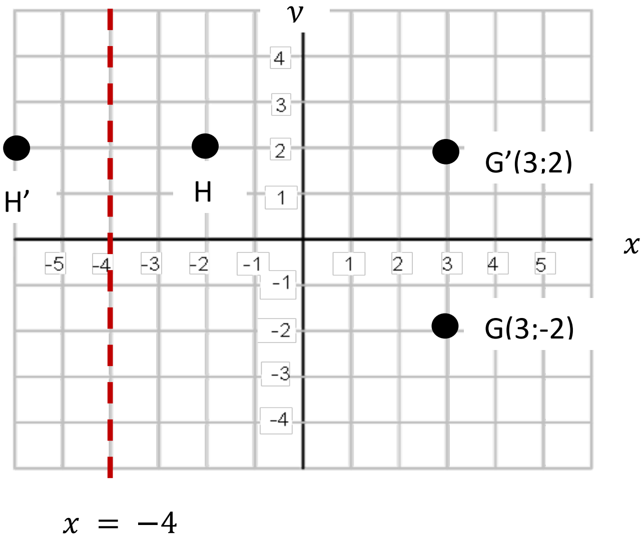
Dot $A(-3;3)$ is the pre-image that is rotated, either through $90°$ clockwise or through 270º anti-clockwise to form the image $A'(3;3)$ with the origin (0;0) as centre of rotation.



Dot $B(-3; 3)$ is the pre-image that is rotated, either through 180º clockwise or through 180º anti-clockwise to form the image $B'(3; -3)$ with the origin (0;0) as centre of rotation.

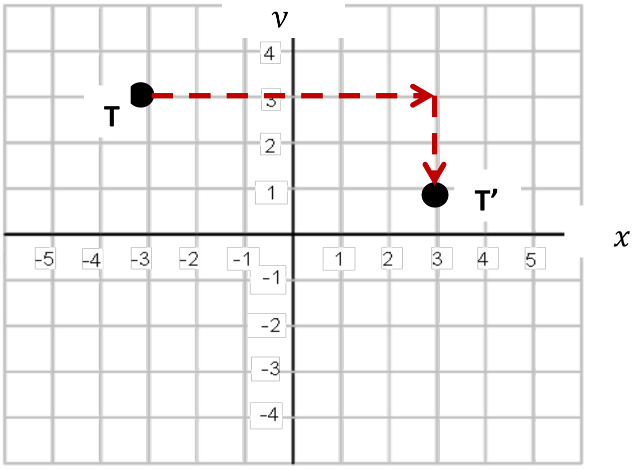
Reflection of a point on the x-axis:

Pre-image $G(3; -2)$ is reflected about the x-axis, resulting in the image $G'(3; 2)$. Point H is reflected about the line $x = -4$. Write down the coordinates of the pre-image H and the image H'.



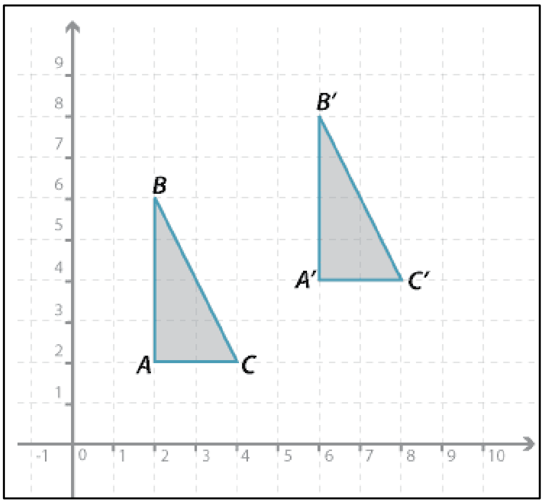
Translation of a point on a Cartesian plane:

Describe the translation path of T to T'as well as their positions or locations

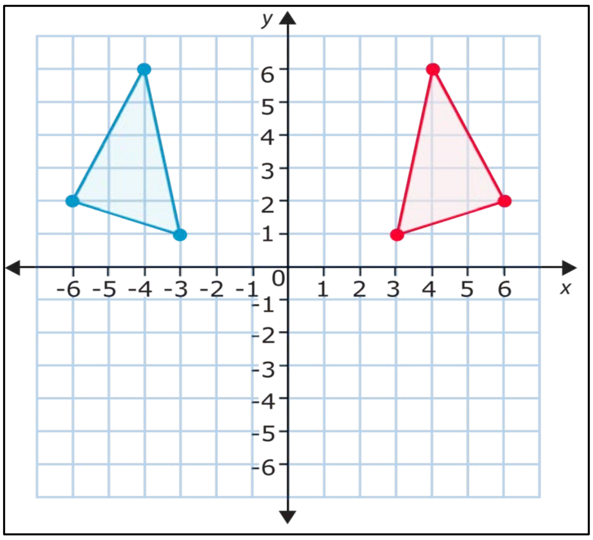


Basic transformations on a Cartesian plane of 2-dimensional shapes

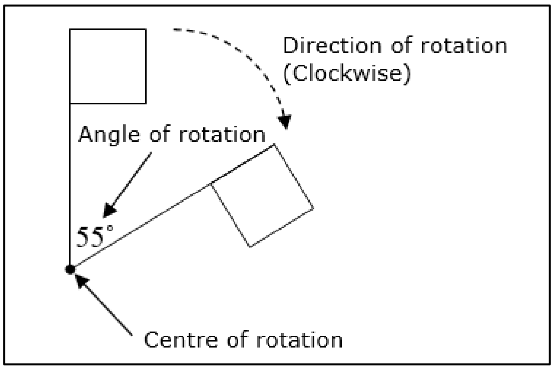
Transformations that conserve size and shape are said to be rigid transformations and show invariance in terms of size and shape. A rigid transformation is also called an isometry, meaning 'same measure' in Greek (Serra, 1997:374). It is important to note that **transformations** can accurately be represented on a Cartesian plane or coordinate system. The ordered pair rule $(x; y) \rightarrow (x + 4; y + 2)$ means moving each point of the original triangular shape (ABC) four units to the right (horizontally) and two units up (vertically) to arrive at the image (A'B'C'). One could also just say, translate by (4; 2) which would have the same meaning as $(x; y) \rightarrow (x + 4; y + 2)$. This would represent a **translation** as shown in the sketch below. The co-ordinates of the points would change from triangle ABC, with vertices A (2; 2), B(2; 6), C(4; 2) to the image A'B'C' with vertices A'(6; 4), B'(6; 8;), C'(8; 4). The prime symbol (') is used in this material to denote an **image** point. The image of a point B under transformations such as reflection, translation or rotation would then be called B' and read as **B prime**.



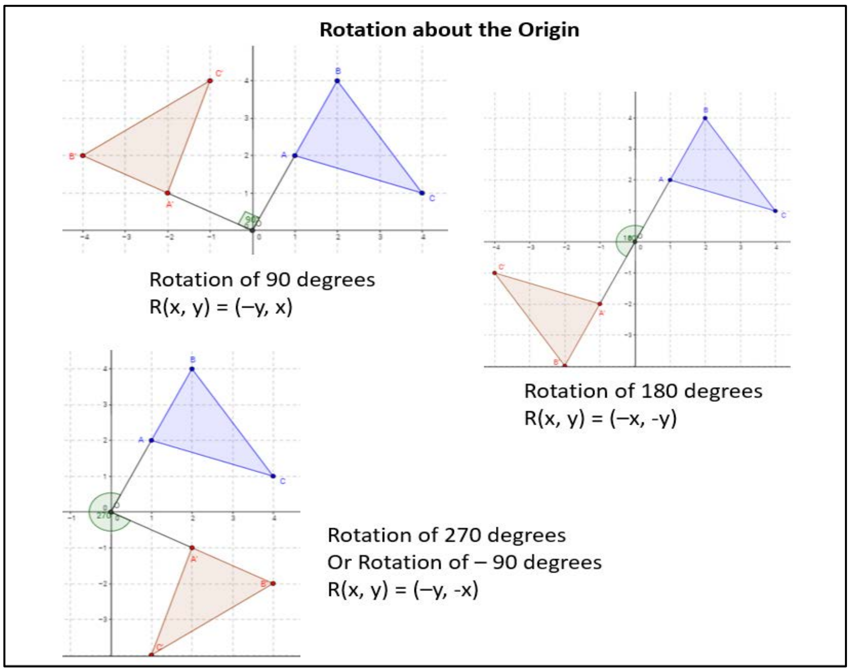
The second type of transformation or isometry we want to illustrate is **reflection**. A reflected image, as in the sketch below, is also called its mirror image. The **line of reflection**, also referred to as the **mirror line** or **axis of symmetry** defines the reflection. A **perpendicular bisector** connects a point (A) on the original shape to the image point (A') on the reflected image.



Rotation is the third type of transformation or isometry. When the figure or shape is rotated all the points in that particular shape rotate through the same number of degrees about a fixed point, referred to as the centre point of rotation.

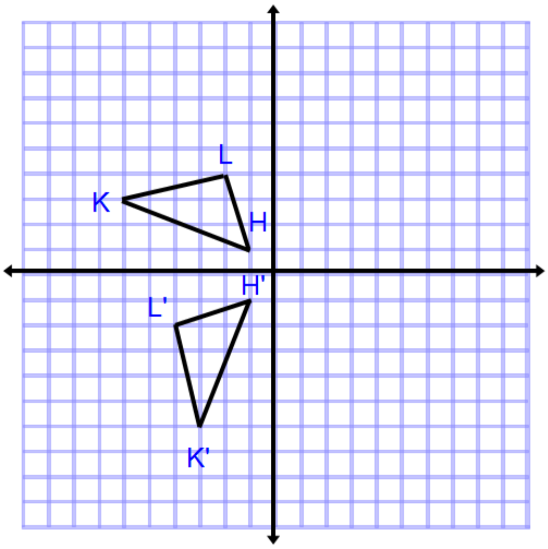


This point is fixed until the shape is rotated. Rotated action is defined by centre point of rotation through a number of degrees (eg. 90º; 180º; 270º; etc) and the direction of the motion (viz. **clockwise** or **anti-clockwise**).



Activity C

1. Study the sketch and answer the questions that follow:



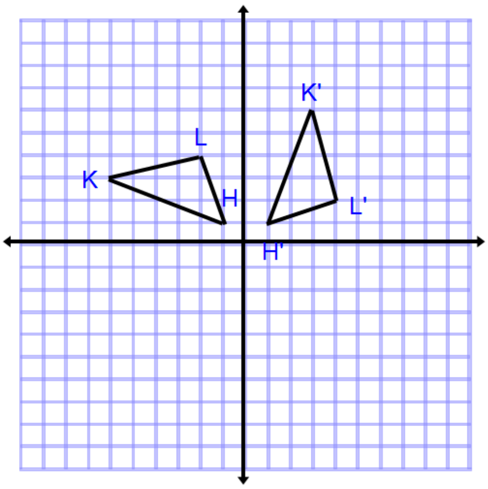
1.1 Describe the rotation of the pre-image to the image centered at (0, 0).

Degree of rotation	Direction of rotation

1.2 What are the coordinates for the pre-image and image?

Pre-image	Image
H (,)	**H' (,)**
L (,)	**L' (,)**
K (,)	**K' (,)**

2. Study the sketch and answer the questions that follow:



2.1 Describe the rotation of the pre-image to the image centered at (0, 0).

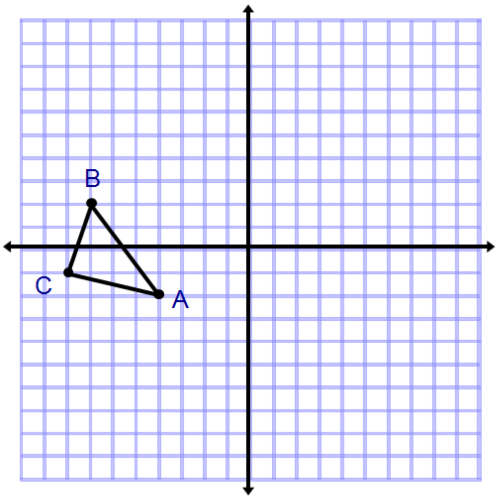
Degree of rotation	Direction of rotation

2.2 What are the coordinates for the pre-image and image?

Pre-image	Image
H (,)	**H' (,)**
L (,)	**L' (,)**
K (,)	**K' (,)**

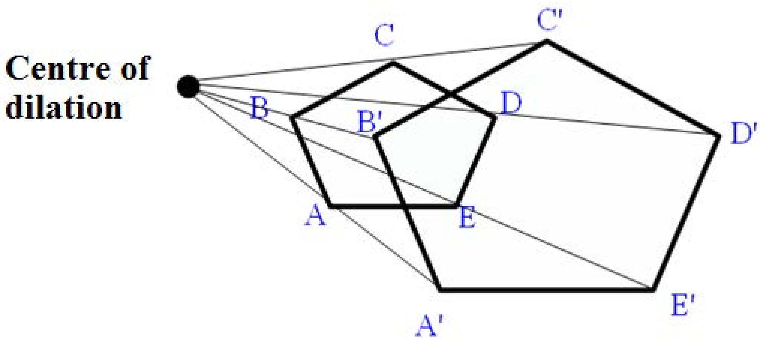
3. Find the coordinates of the images ($A'B'C'$) of triangle ABC for the given rotation about the origin in the graph below

3.1 90° CC 3.2 180° CC 3.3 270° CC



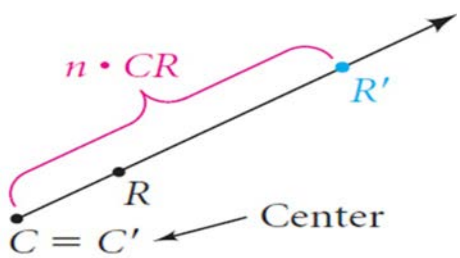
Dilations and Similarity Motions

Any figure mapped by a rigid motion is unchanged in both size and shape. Suppose, however, we wish to find transformations of the plane that preserve shape but change the size of figures. Transformations that change size but not shape are quite common: Consider making an enlargement or reduction on a photocopy machine or using the Zoom command from the View menu of a computer program. The simplest transformation to change the size of a figure, but preserve its shape and orientation, is a dilation (also called a size transformation). Dilations change size but, like translations, produce images in which corresponding line segments are parallel. This property allows you to determine both the centre and the scale factor of the dilation. Simply draw some lines through pairs of corresponding points in the image and pre-image figures; the lines will intersect at the centre O of the dilation (Long et al p599). Thus, a **dilation** is a transformation where a figure or shape is **reduced** or **enlarged** by a given **scale factor** with respect to a point called the **centre of dilation**. See the diagram below.



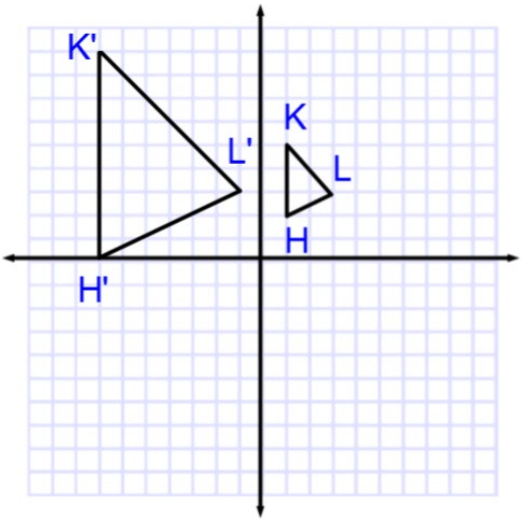
Thus, a dilation with **centre $\boldsymbol{C}$** and a **scale factor of $\boldsymbol{n}$** is a transformation for which the following are true:

1. The image of C is itself.
2. For any point R, R' is on $\overrightarrow{CR}$
3. $CR' = n \cdot CR$



Activity D

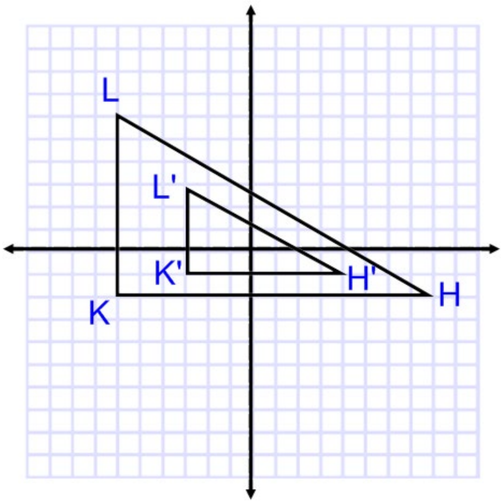
1. Describe the dilation of the pre-image to the image.



1.1 Is this an enlargement or a reduction?
1.2 Write down the co-ordinates of the centre?
1.3 Give the scale factor.
1.4 What are the coordinates for the pre-image and image?

H (,)	**H' (,)**
L (,)	**L' (,)**
K (,)	**K' (,)**

2. Describe the dilation of the pre-image to the image in the following sketch:
2.1 Is this an enlargement or a reduction?
2.2 Write down the co-ordinates of the centre of dilation?
2.3 Give the scale factor of the dilation.
2.4 What are the coordinates for the pre-image and image?

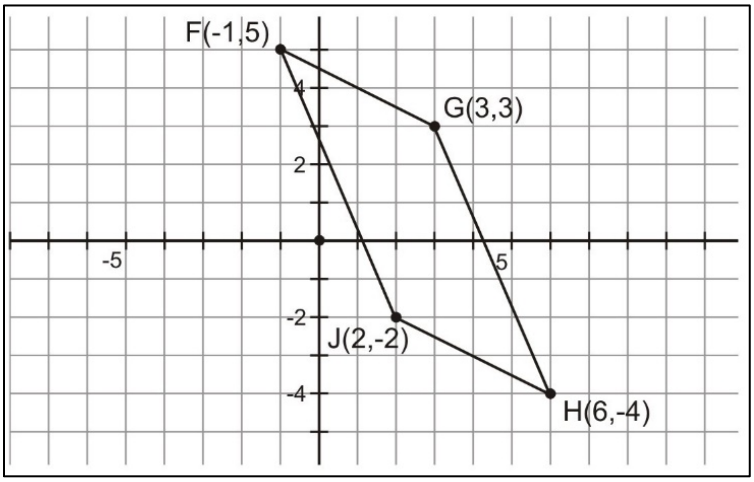


2.5 What are the coordinates for the pre-image and image?

H (,)	**H' (,)**
L (,)	**L' (,)**
K (,)	**K' (,)**

Further Transformation related activities

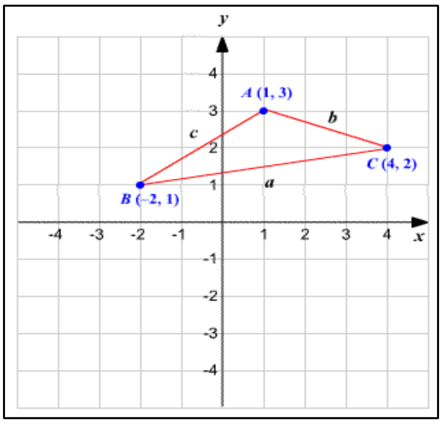
1. Which of the following "transformations" correspond to a rigid motion? Explain the reasoning you have used to give your answer.
   - (a) A deck of cards is shuffled.
   - (b) A completed jigsaw puzzle is taken apart and then put back together.
   - (c) A jigsaw puzzle is taken from the box, assembled, and then placed back in its box.
   - (d) A painting is moved to a new position on the same wall.
   - (e) Bread dough is allowed to rise.
2. Which of the following motions are rigid?
   - (a) A book is moved to a new shelf in a bookcase.
   - (b) A book is opened to page 34.
   - (c) A balloon is inflated.
   - (d) A bowling ball is rolled down the alley.
   - (e) A length of yarn is wound into a ball.
3. David and Desmond were asked to reflect the pattern-block figure shown on the left over the horizontal line through the center of the square. They rearranged the blocks to form the figure on the right.
   - (a) What type of rigid motion did David and Desmond use?
   - (b) How would you help clear up their misunderstanding?
4. Lisa was asked to identify the type of rigid motions used to obtain the following pattern from the bent-arrow motif at the far left:
   She answered, "rotation and glide–reflection."
   - (a) Is Lisa's answer correct?
   - (b) How would you help guide Lisa to the correct answer?
5. A student believes that all rectangles are similar, since all rectangles have four right angles and opposite sides have the same length. How would you respond to this student? In particular, how does the geometrical meaning of "similar" differ from its more common-use meaning?
6. Study the sketch below:
   - (a) Identify the shape in the sketch.
   - (b) Translate the shape by (3;-2), and write the coordinates of the new figure F'G'H'J'.



7. Look at the sketch below.

(a) Identify the triangle type.

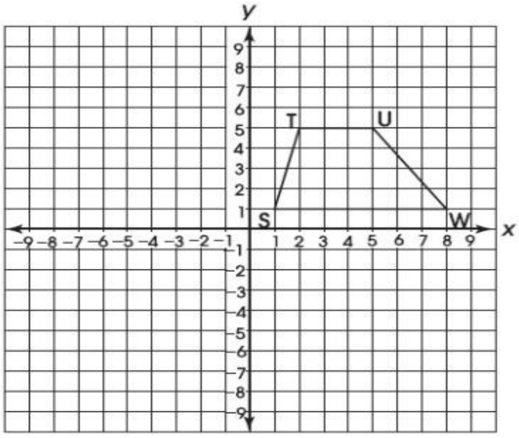
(b) Reflect triangle ABC along the x-axis. Write down the coordinates of the reflected figure $A'B'C'$.



8. Study the sketch below.

(a) Name quadrilateral STUW.

(b) Translate the figure by (x-5;y+1), and write down the points and graph the new figure on the coordinate plane.



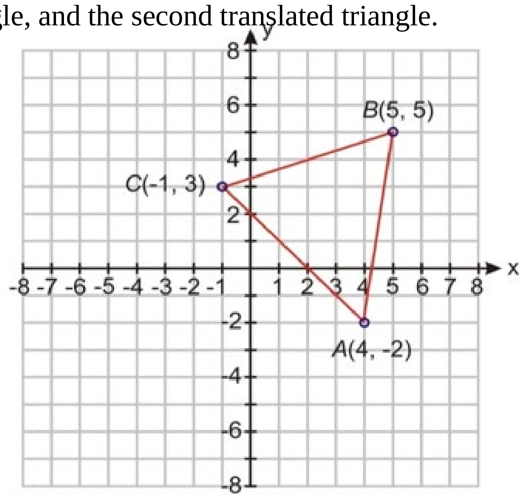
9. Study the triangle below.

(a) Write down the coordinates of each point in the trapezium.

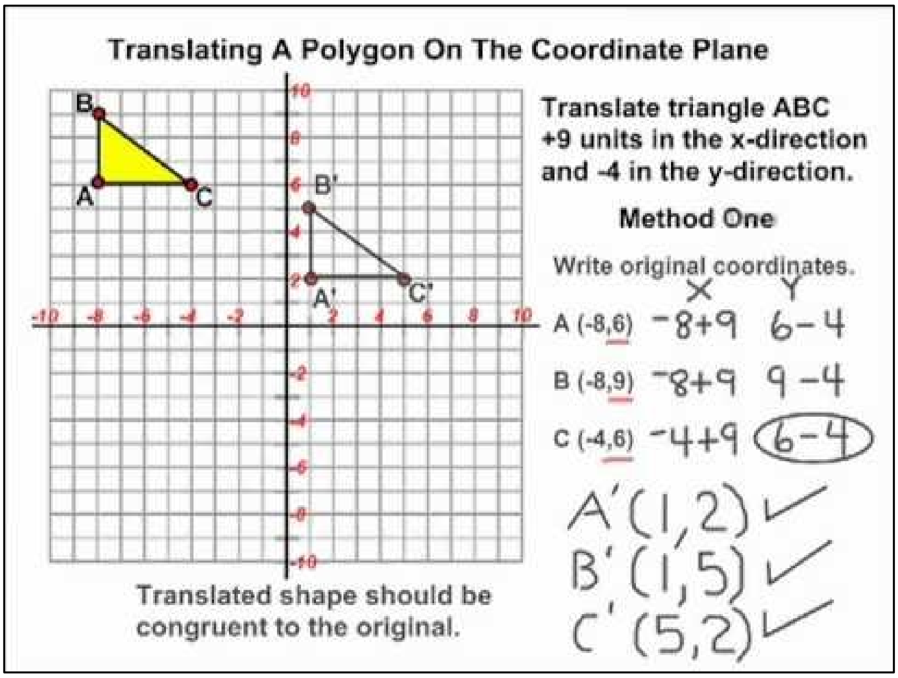
(b) Translate the triangle by $(3;\ -4)$;

(c) Now translate the translated triangle by $(x-5;\ y+3)$

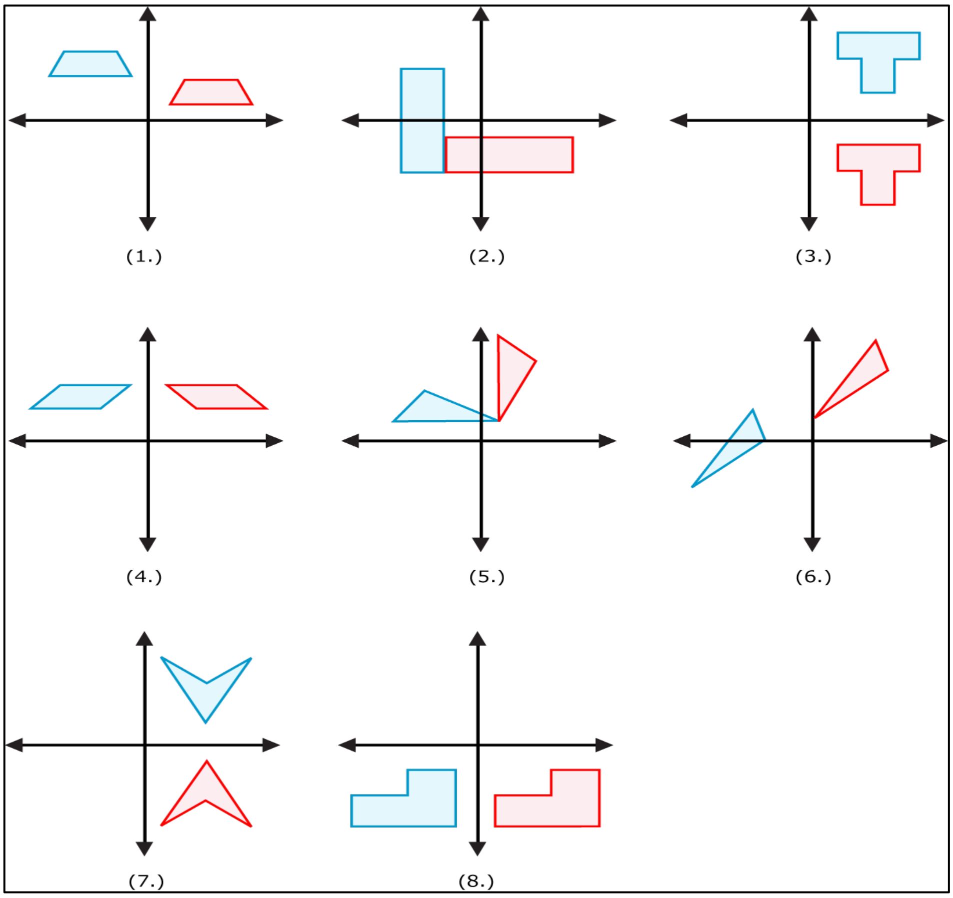
(d) Write down the coordinates of triangle ABC, the original triangle, $A'B'C'$ the first translated triangle, and the second translated triangle.



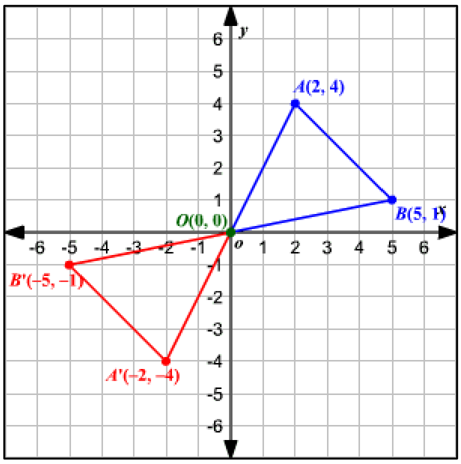
10. Evaluate the following solution and say whether you agree with the learner's efforts.



11. Identify each of the following transformation types.



12. (a) Use a protractor to measure the angle of rotation with centre of rotation at the origin, anti-clockwise in the sketch below.



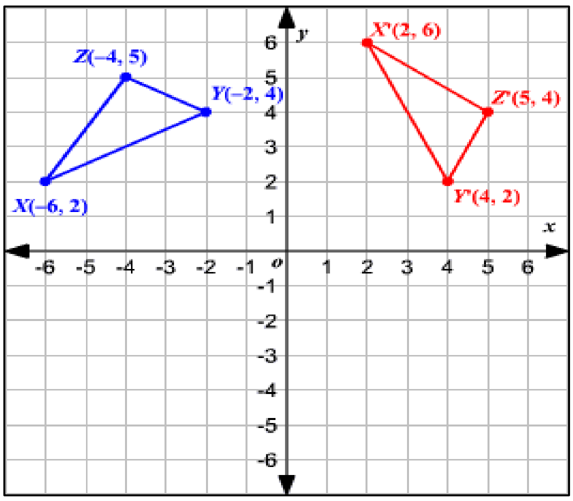
(b) Measure the following angles: $\angle AOA'$; $\angle BOB'$; $\angle AOA'$; $and\ \angle BOB'$.

Write down your findings.

13. Study the following sketch. The centre of rotation in this case is the origin (0; 0).

(a) Connect X and X' or Y and Y' or Z and Z' to the origin and determine the size of the angle by moving in an anti-clockwise direction.

(b) Draw the image of X''Y''Z'', by rotating through 180o in a clockwise direction with the origin as the centre of rotation.



Unit 2	Title of unit: Introducing tessellations

Content standard which this activity addresses

7.2 Drawing and constructing representations of tessellations of two-dimensional geometric shapes or three-dimensional objects using transformations and a variety of tools

7.3 Comparing geometric patterns (tessellations) that share common characteristics (e.g. form, line, angle, vertex arrangement, space)

Intent of this activity

Tessellations: Definition

A tessellation of a flat surface is the tiling of a plane using one or more geometric shapes, called tiles, with no overlaps and no gaps and continues in all directions. In mathematics, tessellations can be generalized to higher dimensions and a variety of geometries.

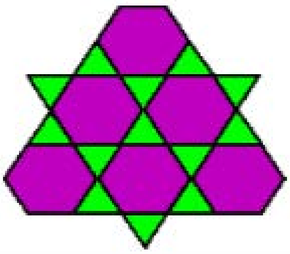
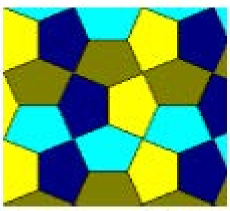
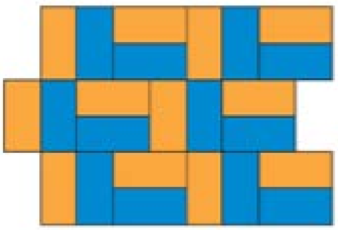
This section explores patterns formed by covering the plane with repeated shapes (or motifs) that neither overlap nor leave uncovered gaps. The art of tiling and decorative patterns has a history as old as civilization itself. In virtually every ancient culture, the artisan's choices of colour and shape were guided as strongly by aesthetic urges as by structural or functional requirements. Imaginative and intricate patterns decorated baskets, pottery, fabrics, wall coverings, and weapons (Long, et al, 2015: p.619).

www.bbc.co.uk/bitesize/ks3/maths/shape_space/transformations1/revision/1/

The creation and exploration of tilings provides an inherently interesting setting for geometric discovery and problem solving in the elementary and middle school classroom. In particular, children enjoy learning how to create their own periodic drawings in the style of the pioneering Dutch artist M. C. Escher (1898–1972). (Long et al,2015: p. 619)

DEFINITION: Tiles

Learning about tessellations, involve pre-knowledge of 2-D shapes **(Which knowledge?)** and transformations. A simple closed curve, together with its interior, is a tile. A set of tiles forms a tiling of a figure if the figure is completely covered by the tiles without overlapping any interior points of the tiles. Tilings are also known as **tessellations,** since the small square tiles in ancient Roman mosaics were called *tesserae* (probably because each tile has four corners and four sides; Greek *tessera*, four) in Latin. (Long et al, 2015: p. 621).

Guidelines for lecture plans/Percentage weightings/Time Management

A. TEACHING AND LEARNING AIMS/OBJECTIVES

Planning teaching for optimal learning, means the general and specific aims for this mathematics content should serve as basis from which to plan.

General aims:

The general aim focuses on communicating effectively using visual, symbolic and/or language skills in various modes (CAPS, 2012).

Specific aims:

- Develop a critical awareness of how mathematical relationships are used in social, environmental, cultural and economic relations
- Develop an appreciation for the beauty and elegance of Mathematics
- A spirit of curiosity and a love for Mathematics
- Recognise that Mathematics is a creative part of human activity

Lesson objectives

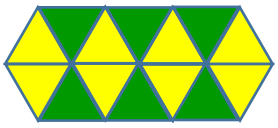
Lesson objectives should be clearly indicated. As can be seen from the objectives below, each includes an action verb to help learners understand what is to be learned, and how it is to be assessed. Learners should know exactly what is expected of them…as such action words such as ‘identify’, ‘describe’, ‘use’, ‘understand’, ‘distinguish’, ‘design’, etc. should feature prominently:

- Identify and describe tiling patterns in real life situations.
- Use 2-D shapes to make geometric patterns and to tessellate (tile) a surface leaving no gaps.
- Understand when a pattern is not a tiling.
- Understand which shapes tessellate and which shapes tessellate only with other shapes.
- Understand the relationship between different topics in mathematics learning & teaching and integration with other subjects.
- Understand the vocabulary/terminology of tessellations.

B. IDEAS FOR MENTAL MATHEMATICS

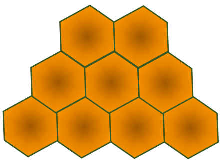
Mental math refers to the practice of doing calculations mentally, that is, all in your head without the use of pencil and paperwork, calculators, or other aids. Mental math usually involves doing basic operations such as addition, subtraction, multiplication and division. Counting forward and backward starting at any value is also used, as well as testing of multiplication tables. The use of mental math activities is viewed as important and generally teachers are expected to start all lessons with some mental activities. Spending ten minutes per day on mental mathematics (as prescribed by CAPS) could be the key to developing learners’ confidence in mathematics. Mental Maths is not only about the instant recall of memorized number facts; it is about developing learners’ own mental images. The teacher should determine how learners do calculations in their minds and compare their mental strategies constructively.

How many angles can you count altogether are in the triangles in the pattern?

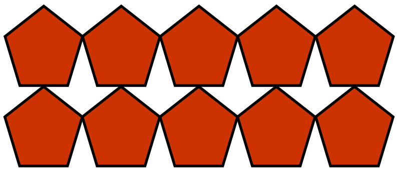


How many white and brown squares there altogether on the chessboard?

How many sides can you count in the hexagons in the pattern altogether? pattern?



How many angles can you count altogether in the pentagons in the pattern?



C. INTRODUCING THE CONCEPT TESSELLATION

An introductory and exploratory exercise to stimulate learners' mentally, is important. A range of questions need to be posed or formulated in such a way that will ultimately ease or guide learners into discovering what it means for shapes or objects to tessellate. Facilitating learning through active questioning should enhance understanding and conceptualisation.

What are the people in the pictures doing?

They are tiling the floors. They are fitting tiles on the floor.

They are laying tiles on floor. They are covering the floor with tiles.

What do you notice about the tiles?

They are squares. They look like squares. They are diamonds.

In each picture the colour of the tiles are the same.

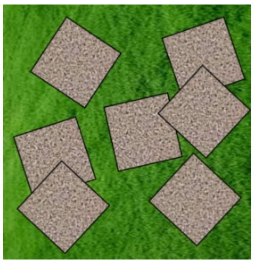
What do you notice about the arrangement of the tiles?

They are next to each other. They are neatly packed. They go up and down.

Which areas or surfaces are normally tiled in people's homes and in our environment?

Walls and floors in kitchens and bathrooms. Pavements in streets.

Floors in shops. Paving around swimming pools, etc.



Mr Adams makes a footpath on his lawn. He uses cement tiles. How is Mr Adam's tiling different from the tiling of the floors in the previous pictures?

It's untidy. It's not arranged properly. There are gaps in between.

Some tiles overlap.

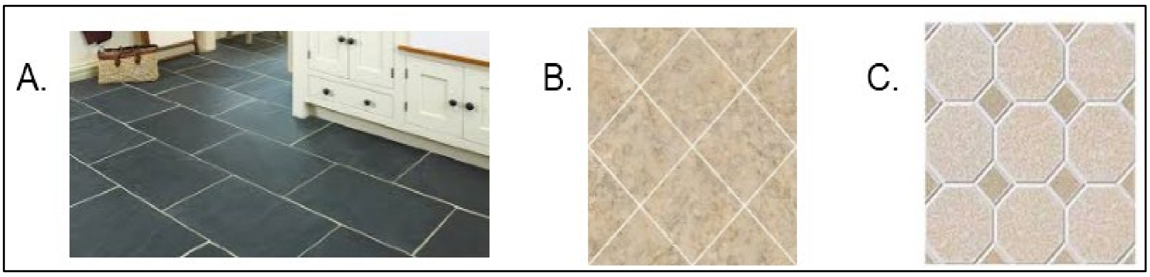
Many schools have traditionally held a transmissionist or instructionist model in which a teacher or lecturer ‘transmits’ information to students. In contrast, Vygotsky’s (Verenikina, 2003) theory promotes learning contexts in which students play an active role in learning. Roles of the teacher and student are therefore shifted, as a teacher should collaborate with his or her students in order to help facilitate meaning construction in students. Learning therefore becomes a reciprocal experience for the students and teacher.

The patterns in the first two pictures are called TESSELLATIONS.
The pattern that Mr Adams created with the tiles is a NON-TESSELLATION.
How do you think we could describe a TESSELLATION?

Any rigid motion will be shown to be one of just four basic types: a translation, a rotation, a reflection, or a glide–reflection. In elementary school terminology, the four basic motions are more informally called, respectively, a slide, a turn, a flip, and a glide. The concept of a rigid motion then allows us, in Section 2, to investigate and classify symmetries and patterns in a precise way. In Section 3, interesting and useful patterns are created by repeatedly transforming a figure to new positions by means of rigid motions. With practice, we will even create artistic tilings that are suggestive of the graphic w ork pioneered by M. C. Escher . (Long etal, p587).

Covering a surface with shapes leaving no gaps or overlaps, is called **tiling**. The formal name for tiling is **tessellation**.

Are the arrangements of tiles in the pictures tessellations? Explain.
What is different about the tessellations in the pictures?



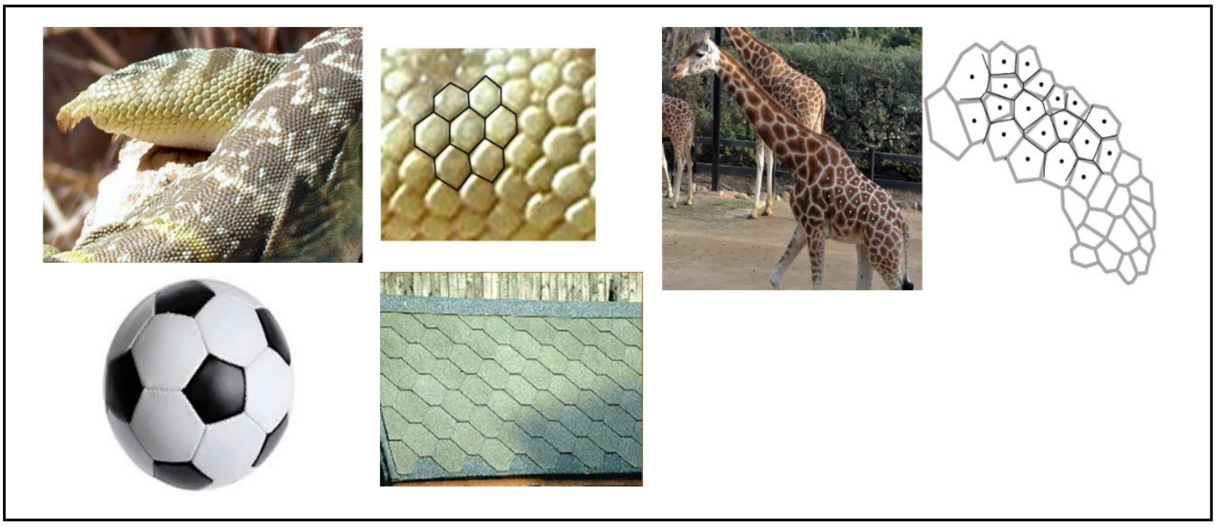
D. SOCIAL KNOWLEDGE TRANSFER

Vygotsky’s sociocultural theory of human learning describes learning of mathematics as a social process and the origination of human intelligence in society or culture. The major theme of Vygotsky’s theoretical framework is that social interaction plays a fundamental role in the development of cognition and mathematical concepts. Vygotsky also believed in the social and cultural nature of development (Verenikina, 2003).

E. MAKING CONNECTIONS

Vygotsky focused on the connections between people and the sociocultural context in which they act and interact in shared experiences. According to Vygotsky, humans use tools that develop from a culture, such as speech and writing, to mediate their social environments. Initially children develop these tools to serve solely as social functions, ways to communicate needs. Vygotsky believed that the internalization of these tools led to higher thinking skills.

Examples of tessellations in our environment



F. CLASS & GROUP ACTIVITIES

"Cooperative learning" (i.e., jigsaw, learning together, group investigation, student teams-achievement divisions, and teams-games-tournaments) is a generic term that is used to describe an instructional arrangement for teaching academic and collaborative skills to small, heterogeneous groups of students (Rich,1993; Sharan,1980). Cooperative learning is deemed highly desirable because of its tendency to reduce peer competition and isolation, and to promote academic achievement and positive interrelationships. A benefit of cooperative learning, therefore, is to provide students with learning disabilities (LD), who have math disabilities and social interaction difficulties, an instructional arrangement that fosters the application and practice of mathematics and collaborative skills within a natural setting (i.e., group activity). Thus, cooperative learning has been used extensively to promote mathematics achievement of students both with and without LD (Slavin, Leavey, & Madden, 1984; Slavin, Madden, & Leavey,1984).

Cooperative learning activities can be used to supplement textbook instruction by providing students with opportunities to practice newly introduced or to review skills and concepts. Teachers can use cooperative learning activities to help students make connections between the concrete and abstract level of instruction through peer interactions and carefully designed activities.

Group discussions (Discourse)

Cooperative learning can be used to promote classroom discourse and oral language development. Wiig and Semel (1984) described mathematics as "conceptually dense." That is, students must understand the language and symbols of mathematics because contextual clues, like those found in reading, are lacking in mathematics.

Work with your group. Explore the patterns in your picture.
Name the shapes in the patterns. Explain how the shapes in the patterns have been transformed.
Is the pattern a tiling or tessellation? Explain.

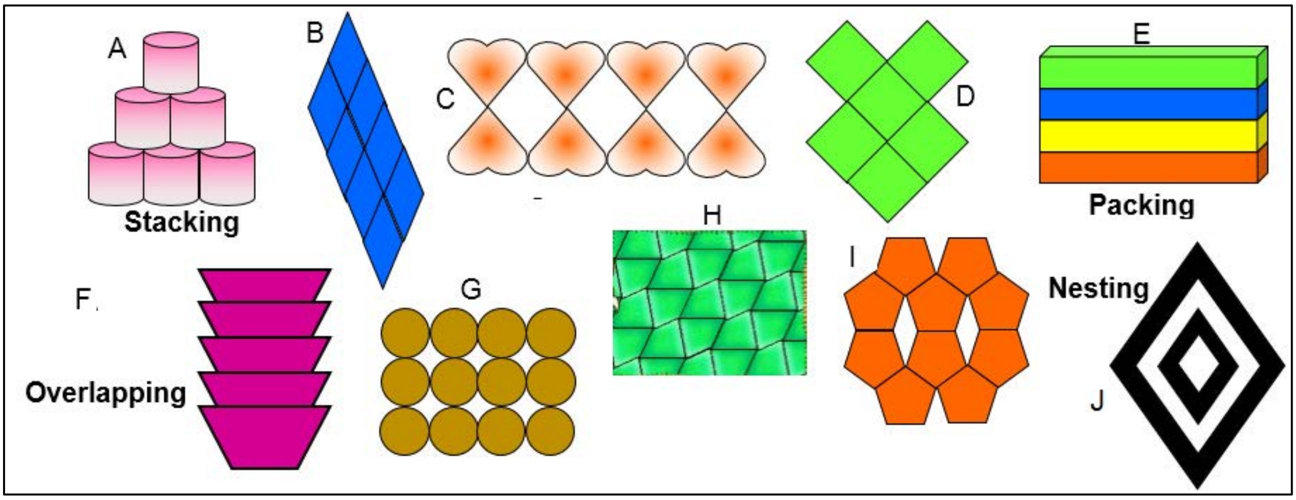
Use of non-examples to enhance logical and critical thinking

According to the National Council of Teachers of Mathematics (NCTM; 1991), learning environments should be created that promote active learning and teaching; classroom discourse; and individual, small-group, and whole-group learning. Cooperative learning is one example of an instructional arrangement that can be used to foster active student learning, which is an important dimension of mathematics learning and highly endorsed by math educators and researchers. Students can be given tasks to discuss, problem solve, and accomplish.

The purpose of the Frayer Model (Frayer, 1969; Buehl, 2001) is to identify and define unfamiliar Mathematical concepts and vocabulary (terminology). Students define a concept/word/term, and describe its essential characteristics, provide examples of the idea and suggest non-examples of the idea (knowing what a concept isn't helps define what it is). This information is placed on a chart that is divided into four sections to provide a visual representation for students. The model prompts students to understand words or terms within the larger context of a reading selection, as it asks students to analyse the concept or word (definition and characteristics) and then synthesise or apply this information by thinking of examples and non-examples. It also activates prior knowledge of a topic and builds connections. Think for instance of four-sided 2-D shapes that are parallelograms and those that are not. Which tessellations are regular and which are non-regular?

Which of these patterns are tessellations? Explain to the class.

Which of the following patterns are not tessellations? Explain.



G. VISUAL MATERIALS

Tessellations as a topic in geometry that has the potential to engage students. Technology can aid in engaging students by showing a variety of examples in different contexts and complexities, providing programs for mess free creation and exploration, as well as integrating the math topic to the world of art. This presentation aims to explore these ideas. The internet provides many resources and programs that can be used to aid in educating students about tessellations. The following activity is an example that provide students information about what tessellations are, how to create them, and where they can be found in the world around them.

Videos to watch to enhance understanding of tessellations

Tessellations Honors Geometry Main Video. YouTube
Tessellation Video Project – Kids' song parody. YouTube
Tessellations in Nature. YouTube

Unit 3	**Title of unit: Practical implementation and applications of transformations and tessellations**

Content standard which this activity addresses

7.4 Demonstrating how (elements and principles) can be used to solve specific spatial visual problems

7.5 Planning and producing works of art applying mathematical techniques, and processes with skill, confidence, and sensitivity.

Intent of this activity

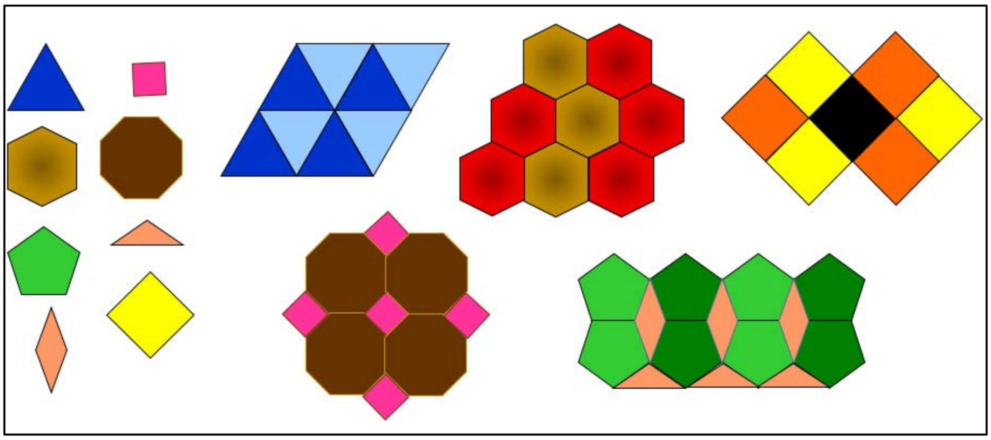
A. PRACTICAL IMPLEMENTATION & APPLICATION

Effective activities which employ skills from various educational disciplines (Science, Technology, Engineering, the Arts, and Math) could propel kids toward inquiry and solution-based knowledge. By combining tessellation explorations with a practical implementation approach to learning, students are challenged to use interdisciplinary and critical thinking skills to investigate how all learning is connected to their world. Tessellations, applications of transformations are connected patterns made of repeating shapes that cover a plane (a 2-D, flat surface that is infinite) completely without overlapping or leaving any holes. A checkerboard is a basic tessellation comprised of alternating coloured squares; the squares meet with no overlapping and can be extended on a surface forever.

Create tessellations. Which shapes did you use? How did you move the shapes to create your pattern or tessellation? Tell the class.

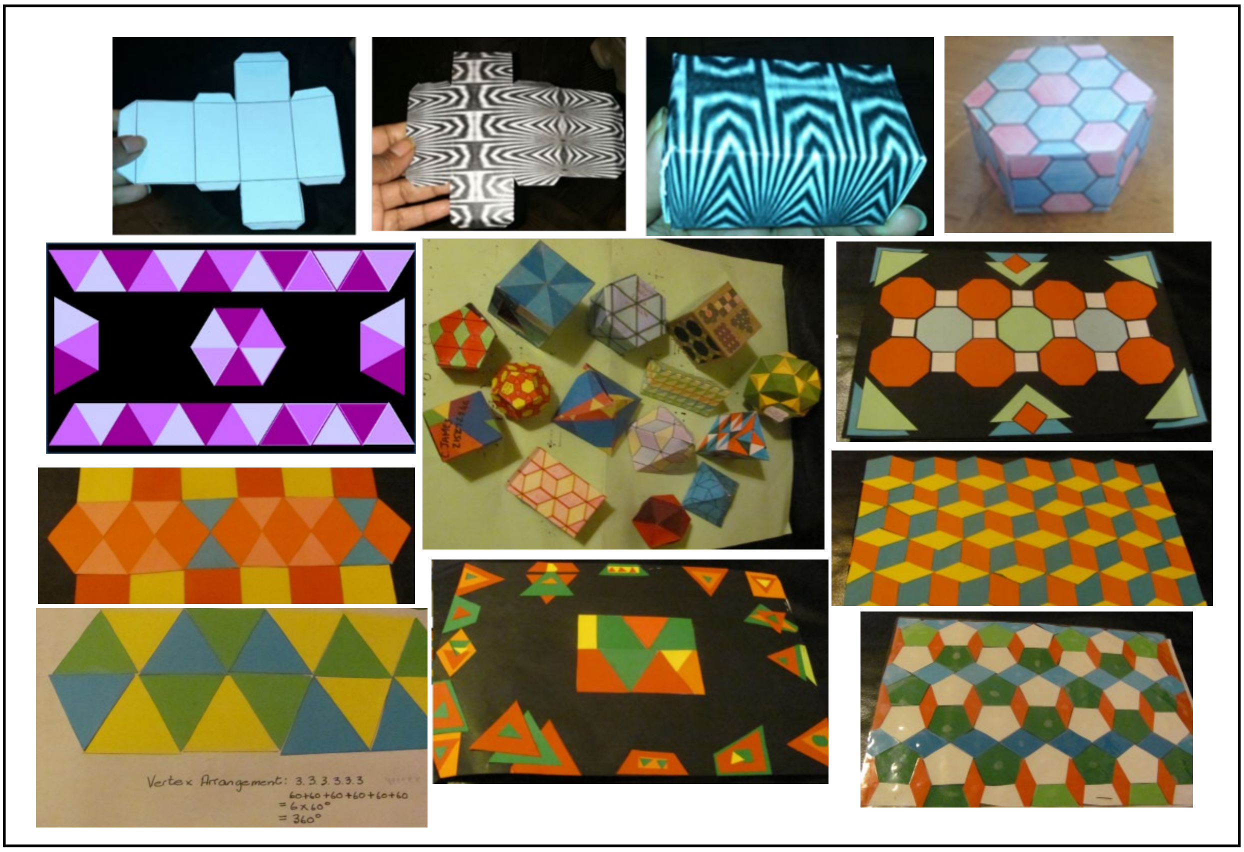
BED 2 – IP2 CPUT Mowbray (2017)

Individual Activities: Using technology – Shapes in Word/PowerPoint or Paint (Purpose?)
Use the single shapes to create tessellations. Use colours you prefer.
Explain to the class how you have created your tessellation.
Name the shapes. How have you transformed the shapes?



B. PROJECT: 2-D SHAPES, 3-D OBJECTS & TESSELLATIONS

Project-based assessment in geometry is an opportunity to utilize and measure the higher order thinking skills of students. A project-based assessment will apply multi-faceted skills to be encompassed into a cumulative project. This can be a singular project at the end of a grading period or it can be done at designated intervals throughout the marking period. The intent is to design the project-based assessment to encompass the lesson plans, teacher worksheets and any additional teacher resources which will ultimately provide a physical example of what was has been learned and what can be applied by the student.



3-D objects and placemats. BED 2 – IP2 Class CPUT (2017)

C. REFLECTION

Becoming a reflective mathematics teacher should be the ultimate, the pinnacle. What does it mean to be reflective and self-critical of your own practice? The ability to reflect on what, why and how teaching responsibilities are performed, the ability to adapt to particular situations and how a teacher develops and grows his or her teaching practice are the qualities that make for a good teacher. Pre-service teachers should be assisted to adopt this disposition of being self-disciplined, self-critical and open to positive and constructive criticism. Posing relevant questions should help to develop an attitude of self-reflection.

What have you learnt about tiling or tessellation this week?
Write a paragraph to describe your learning experiences.

D. VIGNETTES OF STUDENTS' UNDERSTANDING

Example of pre-service teacher's reflection:

1. What have I learnt about transformations?

I have learnt that transformation is movement of objects in coordinate plane, and I've also learnt that there are different types of transformations, which are translation, rotation and reflection. On Translation I have learnt that an object/shape can be shifted to the same direction by the same amount infect you can move or slides a shape. Moreover I've also learnt about reflection, reflection flips a shape over to create a mirror image whether I move it up/down. In addition I also learnt that whenever a shape is reflected, each set of the corresponding must be the same distance from the line of reflection. Lastly I have learnt about rotation that it turns a shape.

2. What am I still unsure of?

I'm still unsure of rotation because even if I saw an object that is reflected I don't understand how it's been reflected. Moreover, I don't know to explain the name of transformation that occurs in the patterns so I can't teach my learners because I'm unsure of it.

3. What have I learnt about tessellations?

Firstly I've learnt a definition of tessellation which says a tessellation is a tiling of plane using one or more 2-D shapes in a repeated pattern with no gaps or overlaps. Moreover I've learnt that u can make tessellation by the polygons, irregular polygons, regular polygons and transformations. I have also learnt that there are different types of tessellation which are regular tessellation, irregular tessellation, Semi regular tessellation and demi regular tessellation. I have also learnt their definitions, regular tessellation it's a tessellation made up of a same regular polygons, irregular tessellation it's an irregular polygons that fit together leaving no gaps, another one is semi regular tessellation its where by you use 2 to 3 polygons with the same vertex arrangement lastly it's a demi regular tessellation its where by you use 2 to 3 polygons with the different vertex arrangement. In addition, I also learnt about the formations that is made with tessellations. Moreover I learnt that a particular tesselation is named by observing the vertex point and ascertaining how many polygons touch the vertex point.

4. What I'm still unsure of regarding tessellations.

Im still unsure of a demi regular tessellation, I don't understand this type of tessellation. Moreover I don't understand the arrangement of shapes, and to name the order of tessellation in each pattern, even in the test I just left the space I didn't respond to the question that was being asked,so this will be difficult for me to teach my learners if as a teacher I don't understand it.
(IP2 – BED2 CPUT student, 2017.)

E. DEVELOPING STUDENTS' CONTENT KNOWLEDGE

Knowledge of students and knowledge of teaching are related to pedagogical content knowledge. Shulman (1995) defines content knowledge as the knowledge about the subject, for example mathematics and its structure. Pedagogical content knowledge is the integration of subject expertise and skilled teaching of that particular subject. It was first developed by Lee Shulman in 1986. Teachers must keep specific methods in mind when preparing to teach a subject. They must combine content with pedagogy. Teachers' understanding of the nature and purpose of the discipline strongly influences their personal pedagogical content knowledge, i.e. what they highlight as important. This means that teachers need to have a sense of what the nature of the discipline is, understanding its organizing concepts as well as its tools.

What should student teachers know about transformation and tessellation?
Transformation:

- A **transformation** is a general term for four specific ways to manipulate the shape of a point, a line, or shape.
- The original shape of the object is called the pre-image and the final shape and position of the object is the image under the **transformation**.
- Types of **transformations in math**. There are four main types of transformations: **translation**, **rotation**, **reflection** and **dilation**. These transformations fall into two categories: rigid transformations that do not change the shape or size of the **pre-image** and non-rigid transformations that change the size but not the shape of the **pre-image**.

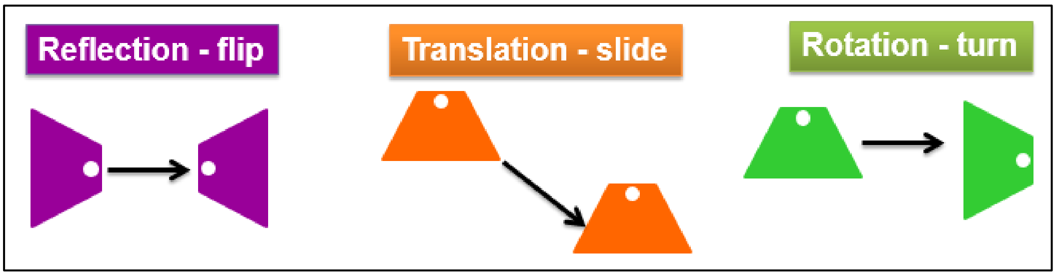
Summary of transformations
Rotation: 90° $(x; y) \rightarrow (-y; x)$
Rotation: 180° $(x; y) \rightarrow (-x; -y)$
Rotation: 270° $(x; y) \rightarrow (-y; x)$
Reflection: $x - axis$ $(x; y) \rightarrow (-y; x)$
Reflection: $y - axis$ $(x; y) \rightarrow (y; -x)$
Reflection: $y = x$ $(x; y) \rightarrow (y; x)$
Point reflection: $(x; y) \rightarrow (-y; x)$
Translation: $(a; b)(x; y) \rightarrow (x + a; y + b)$

Tessellation:

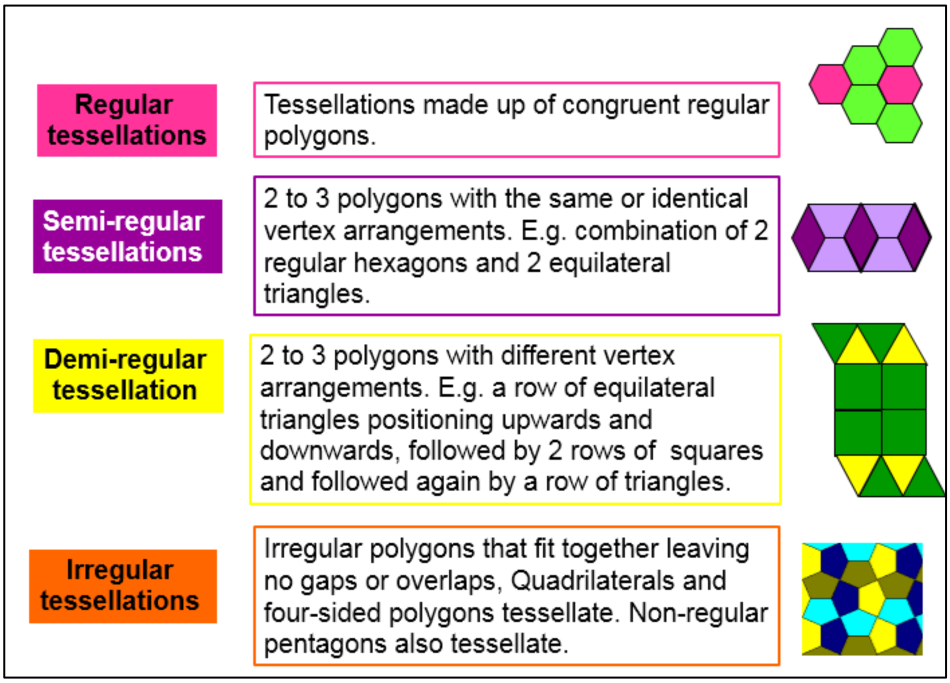
- A tessellation is a tiling of plane using one or more 2-D shapes in a repeated pattern with no gaps or overlaps.
- A regular tessellation is made of a single tile that is a regular polygon, i.e. all sides and angles are congruent.
- Each vertex of a regular tessellation has the same number of tiles meeting at a point, e.g. a checker board.
- A semi-regular tessellation is made of two or more tiles, each a regular polygon.
- At each vertex of a semi-regular tessellation, the same collection of regular comes together in the same order.
- A vertex can be described by the series of shapes meeting at a vertex.
- It is thus possible to design your own artistic and complex tessellations by using transformations or by combining compatible polygons

- The Oxford dictionary describes tessellating as fitting shapes into a pattern without overlapping and with no space in between.
- The term "tessellate" is derived from the Greek word "tesseres," which in English means "four", since the initial tilings were made using square tiles.
- When tessellating shapes, we mainly deal with repeating particular polygons.
- By putting polygons of the same shape and magnitude together, one can investigate which of the shapes tessellate without leaving gaps or overlaps.
- Making tessellations presupposes certain prior knowledge of the following: polygons, regular polygons, irregular polygons, and transformations:

Types of transformations



Types of tessellations

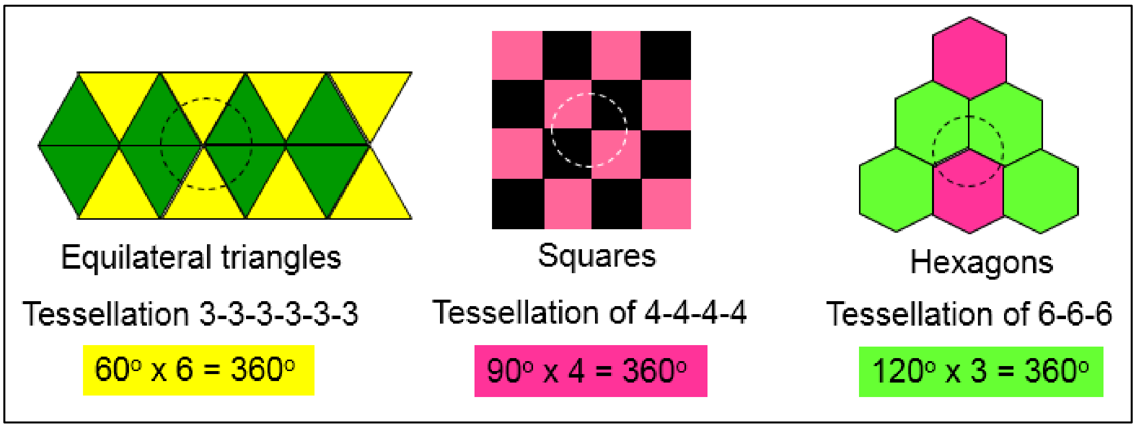


Vertex arrangements

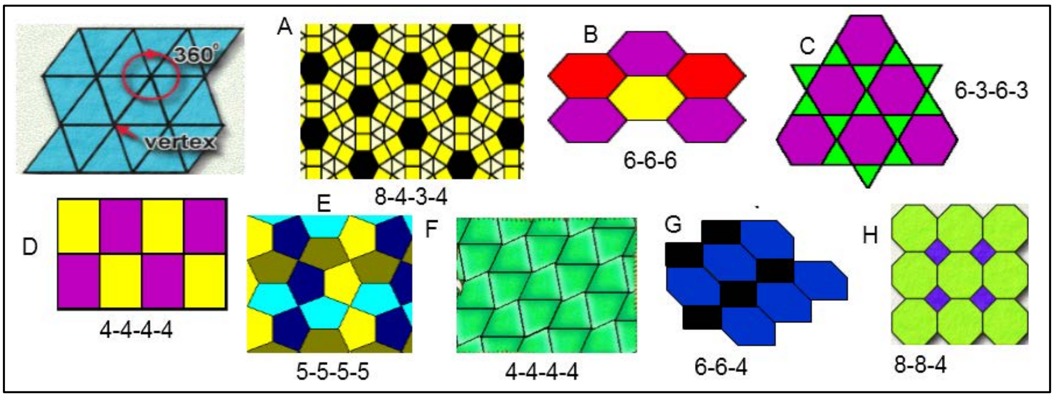
Any arrangement of non-overlapping polygonal tiles surrounding a common vertex is called a vertex figure. Thus, four squares form each vertex figure of the regular square tiling, and three regular hexagons form each vertex figure of the hexagonal tiling. (Long etal 621)

What do you think the formations 4.4.4.4 and 8.8.4 refer to regarding tessellations?

A particular tessellation (arrangement of shapes) is named by observing a vertex point and ascertaining how many polygons touch the vertex point. Conventions are named based on the type of polygons that touch the vertex point. The convention number represents the number of sides of each polygon.



Recognise and name the order of tessellation in each design.



F. EXTENSIONS

Investigation/Assignment

Solving problems is not only a goal of learning mathematics but also a major means of doing so. Some teachers may think that showing learners how to solve a set of problems is the most helpful approach for learners, preventing struggling while saving time. However, it is the struggle that leads to learning, so teachers must resist the natural inclination to take away the struggle. The best way to help learners is to not help too much. Teaching **for** problem solving – in particular modelling and explaining a strategy for how to solve the problem – can actually make learners worse at solving problems and doing mathematics, not better! Learners learn mathematics as a result of solving problems. Mathematical ideas are the outcomes of the problem solving experience rather than elements that must be taught before problem solving (Hiert etal., 1996, 1997). The process of solving problems is completely interwoven with learning; learners are learning mathematics by doing mathematics and by doing mathematics they are learning mathematics (Cai, 2010). Learning by doing, or teaching **through** problem solving, requires a paradigm shift. Teachers change more than just a few things about their teaching; they change their philosophy of how they think learners learn and how they can best help learners to learn (Van de Walle, etal. (2013).

Mathematical investigation refers to the sustained exploration of a mathematical situation. It distinguishes itself from problem solving because it is open-ended. It is a more a divergent activity. Learners might be expected to pose their own problems after initial exploration of the mathematical situation. The exploration of the situation, the formulation of problems and its solutions give opportunity for the development of independent mathematical thinking (Ronda, E, 2010. Curriculum reform in Mathematics Education).

- *Find out if the following regular 2-D shapes tessellate on their own. If not, which shapes do they tessellate with?*
  a) Heptagon *b) Nonagon* *c) Decagon*

- *Which shapes tessellate with a triangle? Investigate.*

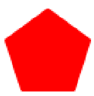
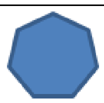
- *Find out who MC Escher was. Go to: Tessellations.org/tess-escher1.shtml. Write a one-page essay.*

TASK: Explore why some regular polygons tessellate and others not.

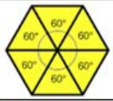
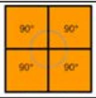
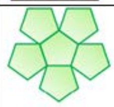
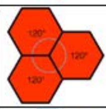
1. Learners are given groups of each of 5 regular polygons (plastic models): triangles, squares, pentagons, hexagons and heptagons.

   a) *Try to cover the plane by putting all the triangles next to one another without overlaps or leaving gaps. Also do this with the other shapes.*

   b) *Use the information found to complete the table below.*
   What have you discovered? Write down all your findings.

Regular shape	Name	Gaps (yes/no)	Overlaps (yes/no)	Tessellates (yes/no)
				
				
				
				
				

2. Complete the table below. Discuss your findings in your groups. Write down your main findings.

Regular shape	Name	No of angles	Size of one angle	Sketch the arrangement of angles around a point	Do angles round a vertex-point add up to 360°	Is the angle size a factor of 360°	Does the shape tessellate? YES/NO
							
							
							
							
							

Analytical rubric for activities

There are basically two main types of assessment rubrics that can be used when assessing students' mathematics performance. In brief, holistic assessment gives learners a single, overall assessment mark for the test or assignment as a whole. Analytic assessment refers to the use of particular criteria (that is, criterion-referenced) and provides learners with a mark for each criterion, and also allows teachers to provide essential feedback on each criterion. The following can be considered before using rubric assessment:

- Make sure that you know what you want to assess;
- Read up on the different types and uses of rubrics;
- Be explicit about the learning objectives;
- Determine the number of dimensions to be used, e.g. 3 or 4, etc;
- Decide where pass/fail is situated on the rubric; and
- Share the criteria and rubric with the class well before the submission date

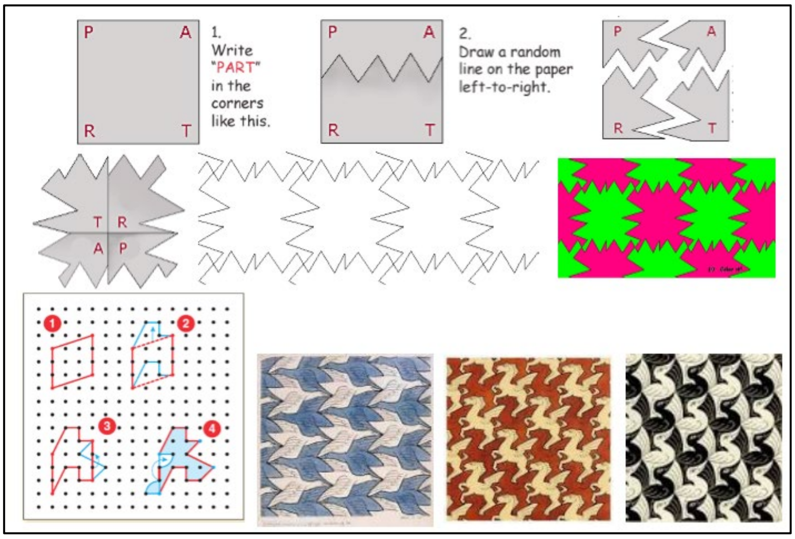
Criteria	**0 – 10**	**11 – 20**	**21 – 30**	**Marks (30)**
Terminology	Incorrect	Partly correct	Accurate use	
Measurement & Calculation of angles	Mostly erroneous	One or two errors and some detail missing	All correct and detailed	
Knowledge of tessellations	Mostly erroneous	Minor gaps in knowledge	All correct and detailed	
Explanation of findings	Hardly any detail; faulty claim or no meaningful statements; discussion reflects major gaps	Some detail; one or two suspect claims; discussion reflects minor gaps	Quite detailed; logical explanations; correct use of terminology; discussion reflects insight and understanding	

G. STUDENT PROJECT/ASSIGNMENT

Create your own tessellations using 2-D shapes & Escher tiles.

Who was M.C. Escher?

Maurits Cornelius Escher (1898 - 1972) is known for his "impossible drawings", drawings using multiple vanishing points, and his "diminishing tessellations". Throughout his various art work, Escher uses complex mathematics, in particular, geometry. In this project we will focus on tessellations... How did Escher use tessellations?



Tessellations are divisions of the plane; more precisely, they are closed shapes that cover the plane. The word is derived from the Latin word "tessella", which were small square stones or tiles used in ancient Roman mosaics. All tessellations can be classified as those that repeat, are non-periodic, quasic-periodic, and those that are fractals. The tessellations that fall in the repeating category can then be classified as being regular (consisting of three types) or semi-regular (consisting of eight types). (Ellen, L. (nd). http://jwilson.coe.uga.edu/EMT668/EMAT6680.2002.Fall/AllenL/MATH%207200/Escher%20Project/Tessellations&Escher/Tessellations&Escher.html

Examples of students' Escher tessellations

BED 2 – IP 2 CPUT Mowbray, 2017.

H. FURTHER READING AND STUDENT ACTIVITIES

Consider the ideas of Ball, Hill & Bass (2005): How well teachers know mathematics is central to their capacity to use materials wisely. Knowing mathematics for teaching demands a kind of depth and detail that goes well beyond what is needed to carry out the algorithm reliably. If we argue for professional knowledge for teaching mathematics, the burden is on us to demonstrate that improving this knowledge also enhances student achievement.

1. Transformations. KS3/95/Ma/Levels 9-10
2. PBS MATHLINE®ESMP: Tessellations WOW!
3. Van de Walle, J.A. etal.(2008). Elementary & Middle School Mathematics (7th ed). Boston
4. MALATI. Module 2: Geometry, Transformations
5. MALATI. Module 3: Polygons
6. Fathauer, RW. Real-World Tessellations. Proceedings of Bridges 2015. Mathematics, Music, Art, Architecture, Culture.

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