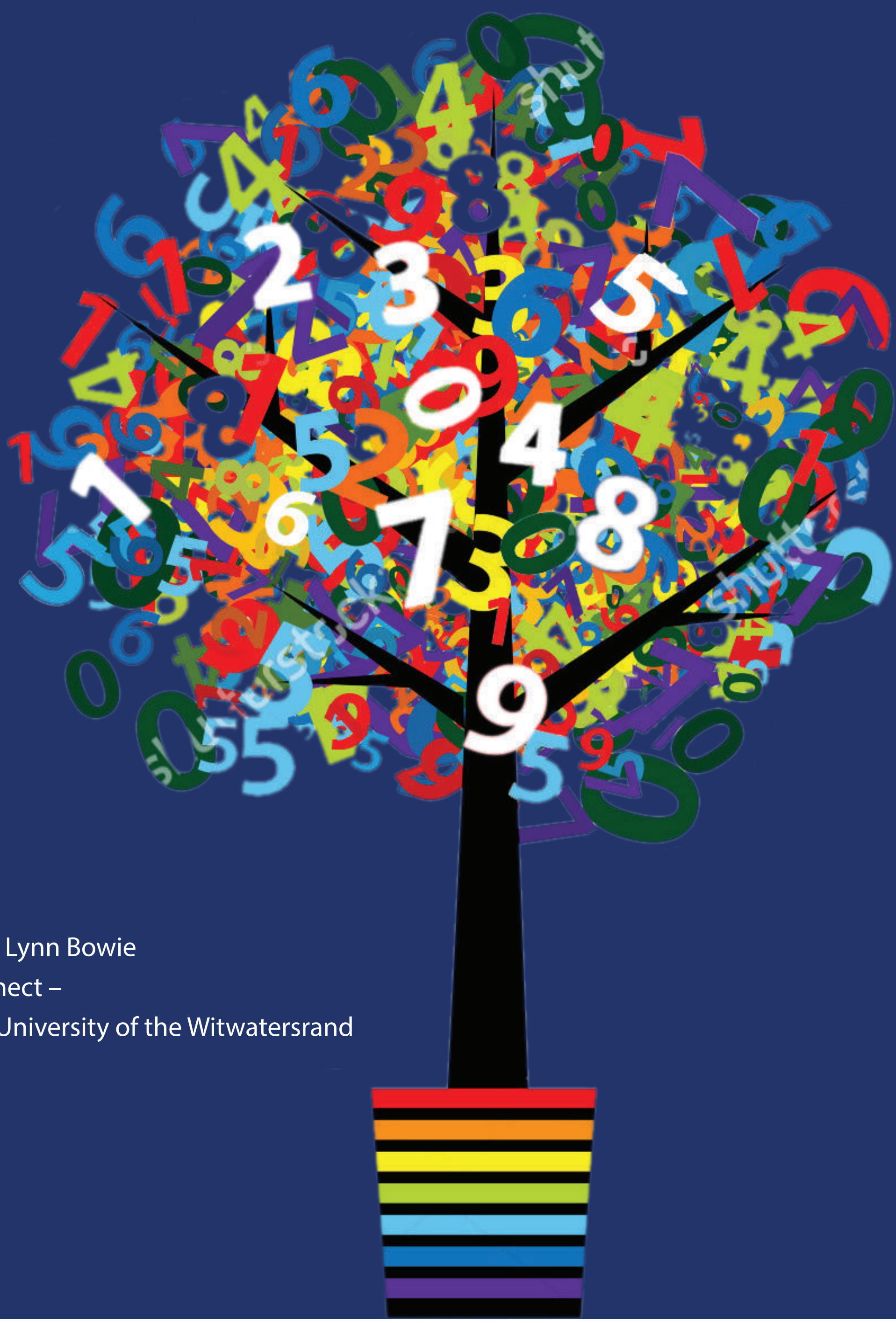
Wits Maths Circles

Teaching ideas and problem-solving tasks for primary mathematics



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theTeacher

Introduction

I'm a magazine junkie. My bedside table looks like the contents came from a doctor's waiting room. I even like the queues at Pick-n-Pay as I can browse through the magazines, and a couple always slip into my basket.

Of course, as Professor of Mathematics Education, I'm also a big fan of mathematics. So, a mathematics magazine? Bliss.

The attraction of magazines includes the pleasure of dipping in during a few spare moments, the delight of being surprised and the satisfaction of marking things to read later ('bookmarking' a PDF is not nearly as satisfying as folding the corner of a page).

Dipping into the pages here, several things piqued my interest that I'm looking forward to reading carefully and working on at leisure. The comment from Aarnout Bromacher (p. 4) about us being too fixated on right answers made me stop and think about where I stand on this. I was still mulling on this when I read the piece 'See things in more than one way' – maybe the issue isn't about right answers, but expecting there to be only one way to get an answer. Food for thought.

The Mandi, Milly and Moses problem (p. 11) intrigued me so much I felt compelled to start on it. Sitting, as I was, in a café meant I didn't have paper and pencil to hand but the tubes of sugar on the table made good stand-ins for bananas. Who says only young learners need to work practically?

And all those pieces on fractions. I don't know why teaching (and learning) fractions is unpopular. In my experience lessons around fractions can really engage everyone in thinking deeply about mathematics and the ideas here provide great starting points for this.

I am sure you will find your own favourites here, and, like me, you'll be grateful to Hamsa and Lynn for putting this magazine together. I'm also sure they would love to hear from you if you would like to share thoughts or anything you have tried out.

Happy reading.

Mike Askew
Distinguished Professor of Mathematics Education,
University of Witwatersrand.

Contents

Maths is about more than just the answer

At one of our workshops, Aarnout Brombacher, who has worked extensively on maths teaching in schools and with maths teachers, said that one of the key insights he had gained from working in many different classrooms over the years was that we are too fixated on children getting the right answer. This sounded a bit strange: surely we want more children to produce correct answers? He used examples to show that if we focus only on learners producing the right answer then we might well ended up in trouble.

The first example he showed was of a little boy doing a long worksheet of 2 – digit additions in columns. For each of the sums the little boy proceeded as follows:

How not to add

Wrong method, right answer

Step one:	Step two:
2 3	2 3
+ 4 6	+ 4 6
6	6 9

Using this method, the little boy got the right answer for every question on the worksheet and both he and his teacher were satisfied that he could do 2 – digit addition. However when Aarnout asked him to calculate 27 + 46, the little boy continued to use his method and so got the following as his answer:

Wrong method, wrong answer

```
   2  7
 + 4  6
   6 13
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Wits Maths Circle Problem set 1:

1. Without working out the answer, can you fill in the missing numbers in this sequence? Each addition has to give the same total:

   27 + 46 = 28 + __ = 29 + __ = 30 + __ = __ + 50

   Some of these additions are easier to work out the answer to than others.

   Which ones are easier?

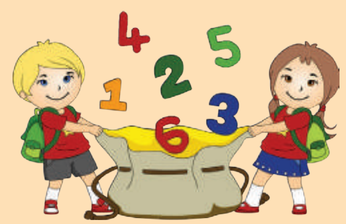
   What makes them easier?

2. Place the digits **1 2 3 4 5 6** into the blocks below so that you get the largest possible sum. (You must use all of the digits)

			+			

Are there **any other ways** you could arrange the digits to get the same largest sum?

Explain clearly how you know that you have created the largest possible sum.



Answers like this point to the importance of focusing on HOW children produce their answers, how they explain their working, as well as the answers themselves. In the problem above it might have been useful to ask the child ROUGHLY how big the answer to 27 and 46 should be. We might want to listen for whether the child can tell us that 27 is a bit less than 30 and 46 is a bit less than 50, so the total when adding the two numbers should be a bit less than 80. The answer should also be more than 20 and 40 put together, which gives 60. So the exact total lies between 60 and 80. In fact, we actually have a bit more than 25 and 45, and these two numbers make 70 together altogether, so the answer lies between 70 and 80.

Several further conversations are possible:

- What should the units digits of the answer be? How do you know that?
- If I want the same overall total, what would i have to add to 28? Or to 30 ? Or to 50?

Stopping at the correct answer and going no further means missing out on a range of further conversations that allow us to check how the child understands the tasks, and how we can extend this understanding. For the child, the risk is that mathematics becomes about performing procedures that are not anchored with any meanings.

Algorithms

Algorithms for long multiplication – whether, what, when and why should we teach them?

We are frequently asked by teachers for help with how to teach long multiplication and long division because learners have difficulty in following and enacting these procedures.

We are asked if there are easier ways that can be used to work out answers that teachers can share.

Our answer to this question is: "It's only easier if learners can make sense of what is going on. That way they can make sense of the steps they have to follow, and figure them out if they forget."

While in school, many of us learnt the algorithm below for working out the answer to long multiplication questions:

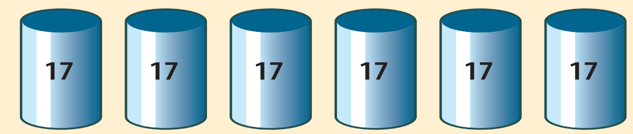
	1	7
×	4	6
1	0	2
6	8	0
7	8	2

Algorithms

Working out answers to long multiplication

There are 17 biscuits inside a packet.
How many biscuits will there be inside six packets?

Young children can imagine this scenario.
They might draw a picture something like this to understand the scenario:

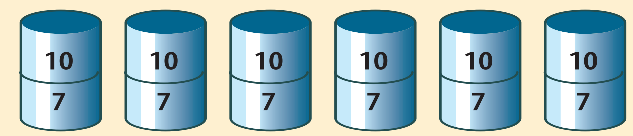


There are several ways to proceed from here. Some children can see that the answer will be:

17 + 17 + 17 + 17 + 17 + 17

Some might say that two packets will have 34 biscuits, so they can work it out as 34 × 3

Some split up the six packets of 17 biscuits into 10 plus seven biscuits in each packet:



As teachers, we can represent this breaking up in a grid:

×	10	7	Totals
6	60	42	102

With practice over time with these diagram and grid models, we can then think about our original problem (17 × 46) in the same way.

×	10	7	Totals
40	400	280	680
6	60	42	102
			782

There are alternative versions of this method, and with experience and familiarity in creating diagram and grid models, children can come up with these alternatives and explore them.

Graphic: JOHN McCANN

Children often have difficulty remembering the order in which numbers are selected for operating on. They make mistakes with "lining up" the numbers in their working out, unsure of when they need to "shuffle up" a column or put in the extra zero.

When we ask teachers how they explain the algorithm, they often recite a set of instructions much like those that we were taught at school: "Start with the units column …. Remember to leave a space before you write the answer … Add these two numbers."

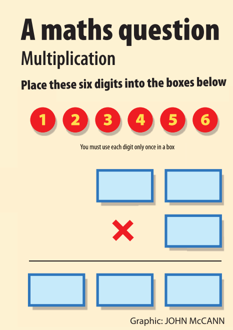
What is striking is the list of things that we have to remember here, and the fact that none of them have any reasons or rationales associated with them. We think there are better ways to think about multiplication that allow learners to think about the situation and how to model it in ways that help them to work out the answer.

In the graphic above we give some examples that lead into calculating 17 x 46.

Wits Maths Circle Problem set 2:

Question

Place the digits 1, 2, 3, 4, 5 and 6 into the boxes below. You must use each digit only once.



Getting a hand on double digits

Children should be fluent with bonds of 10 and adding 10 to a number

Mike Askew is a Professor at Wits University. He is the author of several books for primary maths teachers and parents. In this article we share some ideas that Mike presented at a workshop for Foundation Phase teachers in Johannesburg.

Foundation-phase teaching

Mike proposed that the two things he would like to see children completely fluent with early in the Foundation Phase were:

- Bonds to 10; and
- Adding 10 to a number.

He argued that these skills were crucial building blocks for number sense and that they should be practised every day. He demonstrated the way he would do this in a class, using workshop participants as guinea pigs.

How it works

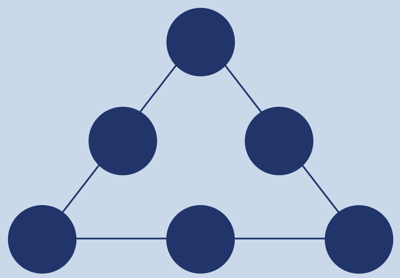
First, he wrote the numbers 53, 17, 72, 81 and 49 on the board. Then he got participants to partner with the person next to them. One person in each pair chose one of the numbers from the board to start with and then the pair, taking turns, counted up from that number in 10s. Everyone managed it successfully. He then got them to repeat the exercise, but this time had them counting up in 10s while clapping their hands, then clapping hands with their partner.

There was much hilarity as many of them made an occasional stumble or stutter as they tried to co-ordinate the two. When he added the final component — one partner getting to control the pace of counting in 10s by either speeding up or slowing down the clapping — there was even more laughter.

Wits Maths Circle Problem set 3:

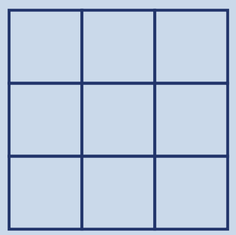
1. Using each of the digits 1, 2, 3, 4, 5, 6 once only, fill in the circles so that the sum of the digits along each of the three sides of the triangle is the same.

Is there more than one way of doing this?



2. Using each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 once only, fill in the small squares so that the sum of each row, column and diagonal of the larger square is the same.

Is there more than one way of doing this?



The purpose

Askew said that this simple (and fun) activity had a number of purposes. The regular practice of counting up in 10s (starting at any number) was important for getting learners entirely fluent in adding 10 to a number. Once children were fairly comfortable doing the counting up in 10s, he added the clapping and the speeding up and slowing down to internalise the counting and make it "automatic".

He argued that no one could coordinate the clapping and counting if they had to think about or work out the next number. Also, the fun element of the activity enabled learners to laugh about their mistakes and created a classroom environment in which mistakes were not shameful, but part of learning.

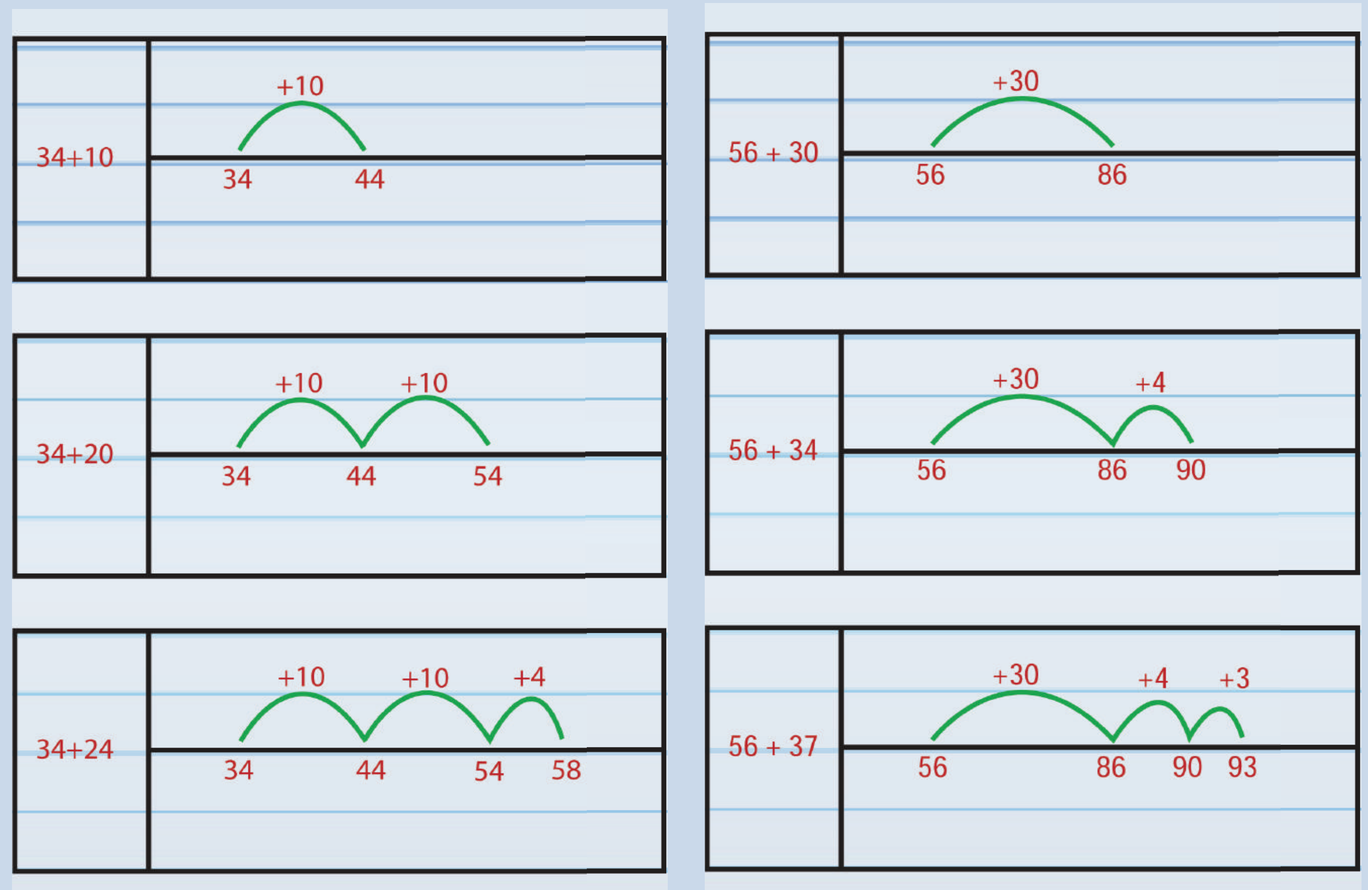
If learners are completely fluent in counting in 10s and their bonds to 10, then Askew's next set of suggested activities enables learners to add two two-digit numbers easily. 'He suggests that learners should be given calculation strings like: 34 + 10, 34 + 20, 34 + 23 as a set, and asked to say the answers to the first two as a way of building up to the answer to the third problem.

There are more examples of calculation strings, using number lines to show the jumping in tens, on the next page.

Internalising mathematics: Examples of calculation strings

Below are two typical calculation strings, along with what we would want the learner to write on the empty number line to represent them:

With practice, children learn to group the multiple 10s together into one jump, building efficiency, as shown below. They can also break down units to work through 10s, rather than having to add in ones — as shown below:



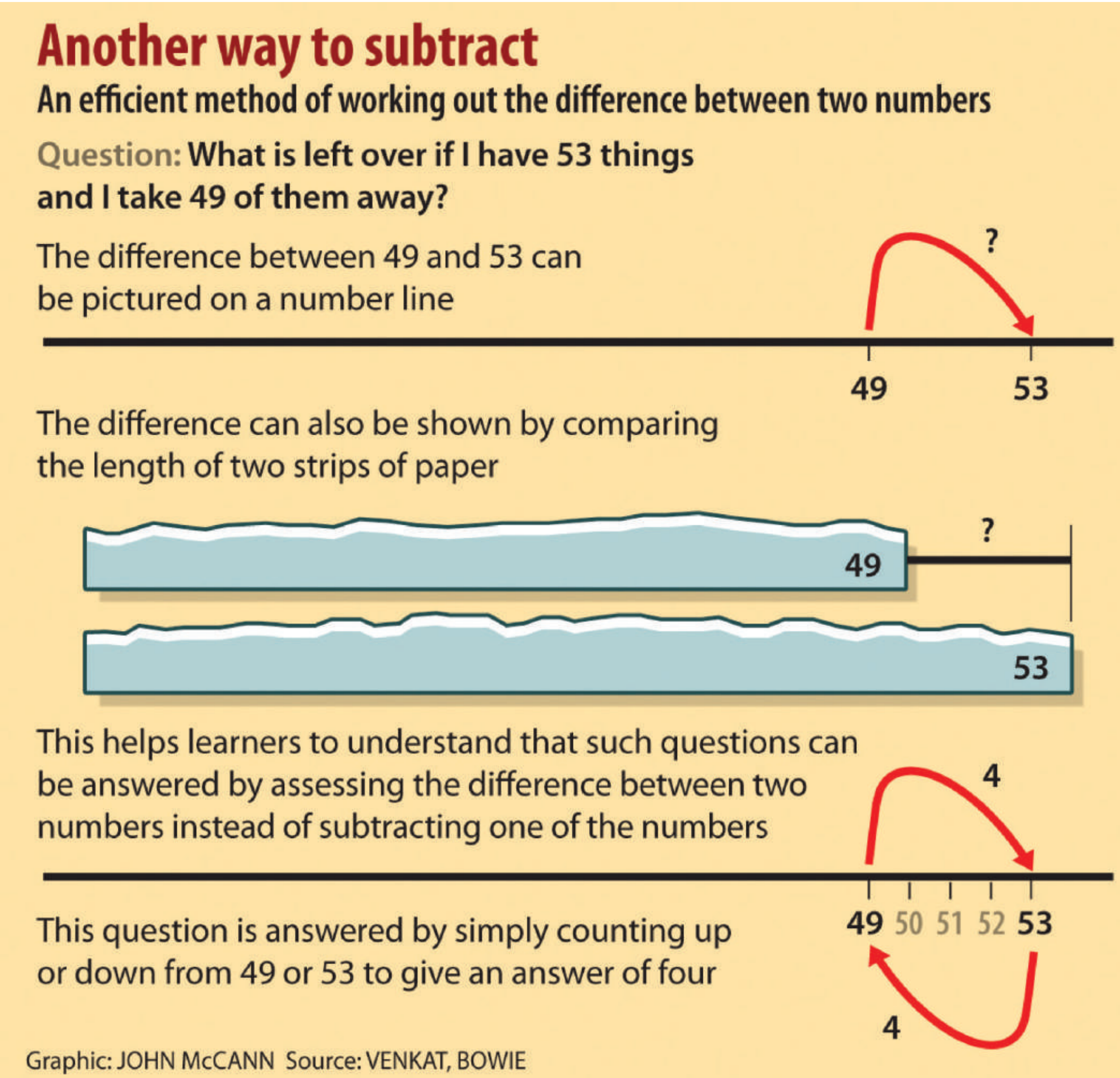
In these calculation strings, the first number is kept whole and the second number is split up. To do 56+37, we calculate 56+30+4+3. In schools it is common to teach learners to split both numbers up into 10s and units (50+6+30+4). Mike Askew said that keeping the first number whole provided a better strategy for mental calculations, as it extended easily to subtraction. For example, 54–36 is easy to do as 54–30=24 and then 24–6=20-2=18. But if we split both numbers, we get 50+4–(30+6) and children get confused over whether this is 50–30+4+6 or 50–30 and 4–6, or others.

Practice makes perfect

Askew said he had observed over and over again that, if learners were given enough of these calculation strings to practise regularly, the ideas became automatic. A learner faced with calculating 35+24 would be able to calculate first 35+20=55 and then 55+4=59 in their heads to reach the answer. Faced with 27+25, they would be able to do 27+20=47 and then see 47+5 as 47+3(+2)=50+2=52. After Askew took participants in the workshop through a number of calculation strings, he gave them some to do mentally.

He asked them to calculate 64 + 28 in their heads and to show a thumbs up near their chest as soon as they had the answer. This is a nice way for a teacher to get an idea of how everyone is doing without the fastest distracting or discouraging the others. In the workshop of more than 30 teachers, one of the fastest mental calculators was one of the teachers' 11-year-old daughter.

See things in more than one way



We were recently working with a group of grade 3s on subtraction. The first question we asked them to do was to calculate 9 – 4. Most of the group happily put 9 fingers in the air and then took away 4 of them to produce the answer 5. We then proceeded to ask them if they could do 53 – 49. For some, the size of the numbers and the fact that they didn't have 53 fingers to work from, meant they couldn't start. A couple knew how to do column subtraction but then proceeded as follows:

- I have 9 fingers and I take 4 away.
- I have 53 things and I take 49 of them away.

Part of the difficulty here is that these learners were only able to think of subtraction as taking away:

But in the case of 53 – 49 it is much easier if you see it as asking you what the difference is between 49 and 53 and can picture it on the number line. Or even just as a comparison between the lengths of two strips of paper: With these diagrams, some learners choose to 'count down from' 53 until they reach 49 – on their fingers or in their heads: '52, 51, 50, 49. There are 4 fingers here, so the answer is 4.' Other children 'count up to' 53 from 49 – again, on their fingers or in their heads. '50, 51, 52, 53. I counted four numbers, so the answer is 4.' The count is simpler and shorter, so as well as being more efficient than trying to take away 49, there is much less chance of the child making a mistake.

Wits Maths Circle Problem set 4:

1. Cut out the digits in the blocks below:

1	2	3	4	5	6

Place the digits into the blocks below so that you get the largest possible difference.

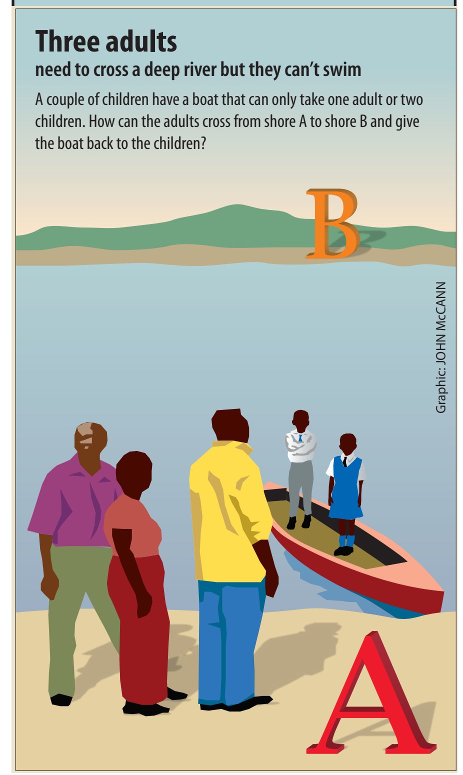
			—			

Explain clearly how you know that you have created the largest possible difference.

2. A logic puzzle:

Look at the 'Three adults' problem in the caption at the side. See if you can solve it and explain your thinking.

Three adults need to cross a deep river, but none of them can swim. They meet two children who have a small boat. However, the boat is so small that it can only support two children or one adult. If a child and an adult get into the boat together, it will sink. How can all three adults use the boat to cross safely to the other side of the river — and return the boat to the two children?



What teachers tell us

Some teachers tell us that children get confused if they provide more than one way of seeing subtraction, so they prefer to explain one method. It may well be that when we introduce subtraction we do it with the 'take away' model, and help learners to produce actions with counters or their fingers, and then diagrams that represent these actions.

In the foundation phase

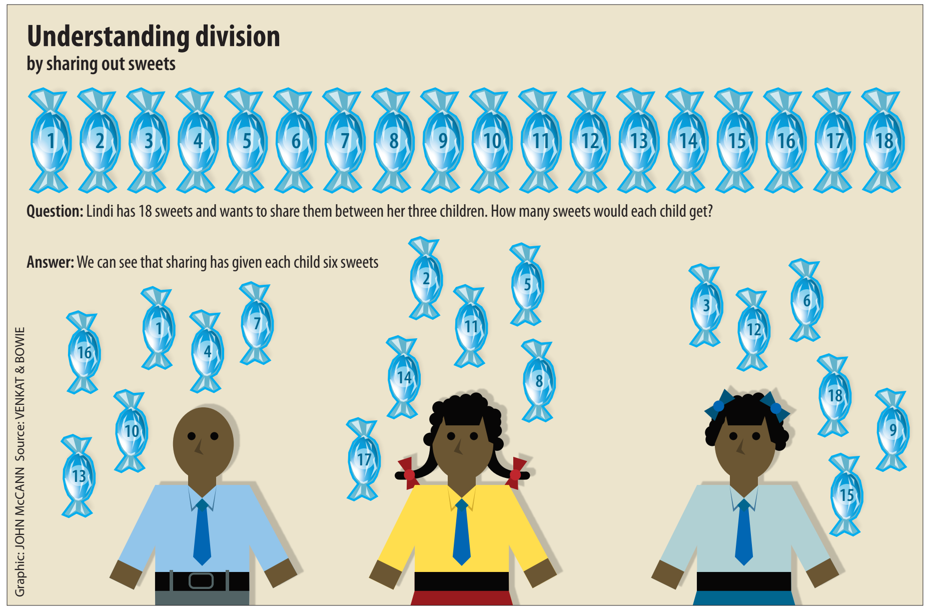
But in the foundation phase, we do want children to be able to answer questions like 53 – 49. And for this kind of example, where the number we are taking away is almost as big as the number we are starting with, being able to see subtraction as 'difference' provides a more effective and a more efficient strategy for working out the answer.

As teachers, we need to work to expand the range of problems that our learners can deal with, rather than limiting them to problems within a small number range that can be worked out by taking away. If children have a flexible understanding of subtraction and can interpret it as either 'take away' or as 'difference' they will be far more able to solve a greater variety of problems accurately and efficiently.

Different ways to divide

In the previous article we talked about why it is often useful to see mathematical concepts in more than one way. We gave examples related to being able to view subtraction as both "taking away" the second number from the first number and also as the "difference between" the two numbers in the number sentence.

In this article, we will do the same thing with division. When we ask learners what they understand by division, they correctly say "sharing". Here is a number sentence in which the action of sharing represents the division operation:



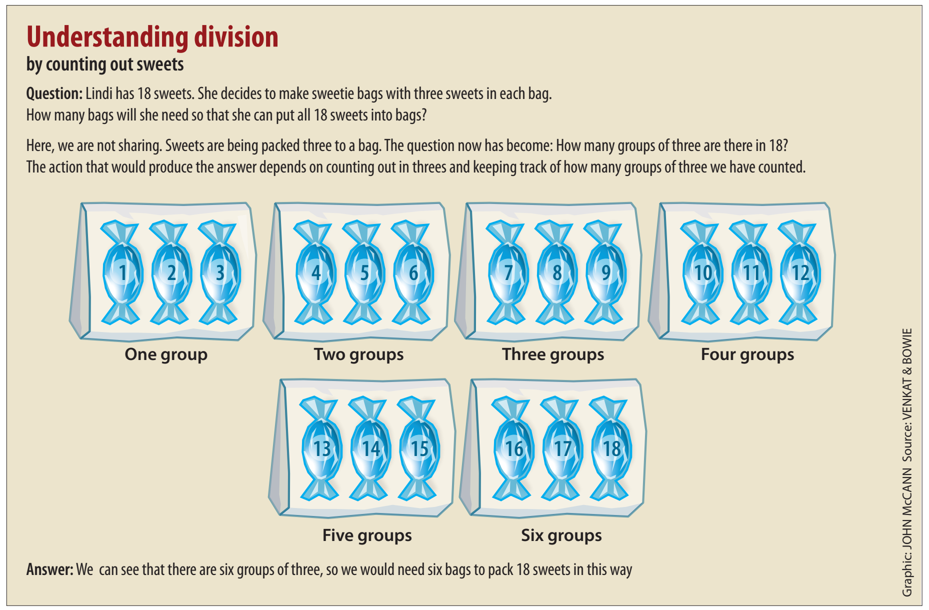
Giving sweets

"Lindi has 18 sweets and wants to share them between her three children. How many sweets would each child get?" The action that would produce the answer to this question is based on Lindi giving one sweet to child 1, then one to child 2, then one to child 3, then another to child 1, then child 2, then child 3, and so on, until she has shared out all 18 sweets.

The diagram on the right represents this sharing action. As the diagram shows, we can see that sharing has given each child 6 sweets, so this is the answer. Each of Lindi's children would receive 6 sweets.

But now let us think about this problem: "Lindi has 18 sweets. She decides to make sweetie bags with 3 sweets in each bag. How many bags will she need so that she can put all 18 sweets into bags?"

Here, we are not sharing. Sweets are being packed, 3 to a bag. The question now has become: "How many groups of 3 are there in 18?"



Counting out

The action that would produce the answer now depends on counting out in 3s and keeping track of how many groups of 3 we have counted. The picture that represents this action would look like this in the second diagram on this page.

Here we can see that there are 6 groups of 3, so Lindi would need 6 bags to pack 18 sweets in this way. Sometimes, teachers say: "The answer is the same, so it doesn't matter if we only teach sharing."

But we think it does matter — and for two reasons. Firstly, in the grouping problem above, children who solve it by sharing often answer '6 sweets' instead of '6 bags'. This alerts us to the fact that they are not really making sense of the problem.

Secondly, being able to think about a division problem like: 42 ÷ 7 in terms of "How many 7s are there in 42?" as well as "42 objects shared between 7 children", brings a useful flexibility. If I can see 42 ÷ 7 as "how many 7s in 42", I can then count in multiples of 7 on my fingers, see that I am holding up 6 fingers, and give the answer 6. A further gain from being able to work with grouping models as well as sharing models comes when we move to questions like this one: What is 9 ÷ 1/2?

Here it does not make sense to say: "I'm going to share 9 objects among half a person". However, it does make sense to ask: "How many halves in 9?" This also allows me to see much more easily that the answer has to be 18, because there are two halves in each whole, and I have 9 wholes.

As we said in our last article, it is tempting to think that showing learners just one way of doing something makes it simpler and less confusing for them. But we think it is important that teachers open up both "sharing" and "grouping" as ways of working with division, if we are going to give our children tools to work with a wider range of problems.

Wits Maths Circle Problem Set 5:

1. Three monkeys, Mandi, Milly and Moses, collected a huge pile of bananas that they were going to share. They put all the bananas together in a big pile and went to sleep. During the night Mandi woke up hungry. She ate one banana and divided the rest of the bananas into three equal piles. She took one of the piles off into the forest to hide them away for herself.

   A little while late Milly woke up hungry. She ate one banana and divided the rest of the bananas into three equal piles. She took one of the piles off into the forest to hide them away for herself.

   And just before morning Moses woke up hungry. He ate one banana and divided the rest of the bananas into three equal piles. He took one of the piles off into the forest to hide them away for himself.

   When all the monkeys got up in the morning they were shocked to see only 6 bananas left. They all denied taking any bananas during the night. Eventually they shared the 6 bananas equally between them and headed off into the forest angry with each other.

   a) How many bananas were in the pile when the monkeys went to bed?
   b) Did each monkey end up getting an equal number of bananas?

2. We are told that 5542 ÷ 17 = 326. Use this fact to say what 326 x 18 will be without carrying out the multiplication.

Supporting efficient problem solving

Supporting teachers to model efficient problem-solving

Evidence suggests that children in South Africa are not making enough progress in learning numeracy and mathematics during primary school. One example of this lack of progress is that some children seem unable to move beyond very basic methods for solving number problems.

Supporting teachers

Our work at the University of the Witwatersrand is concerned with supporting teaching. One way to support children to develop more sophisticated and efficient problem solving skills is to support teachers to model efficient strategies rather than merely accept learners' own strategies.

We see learners' strategies as the starting point for teaching but that learning means being able to move beyond the strategies that learners bring to a lesson. In this article, we provide an example of what a more sophisticated model might look like.

Foundation Phase

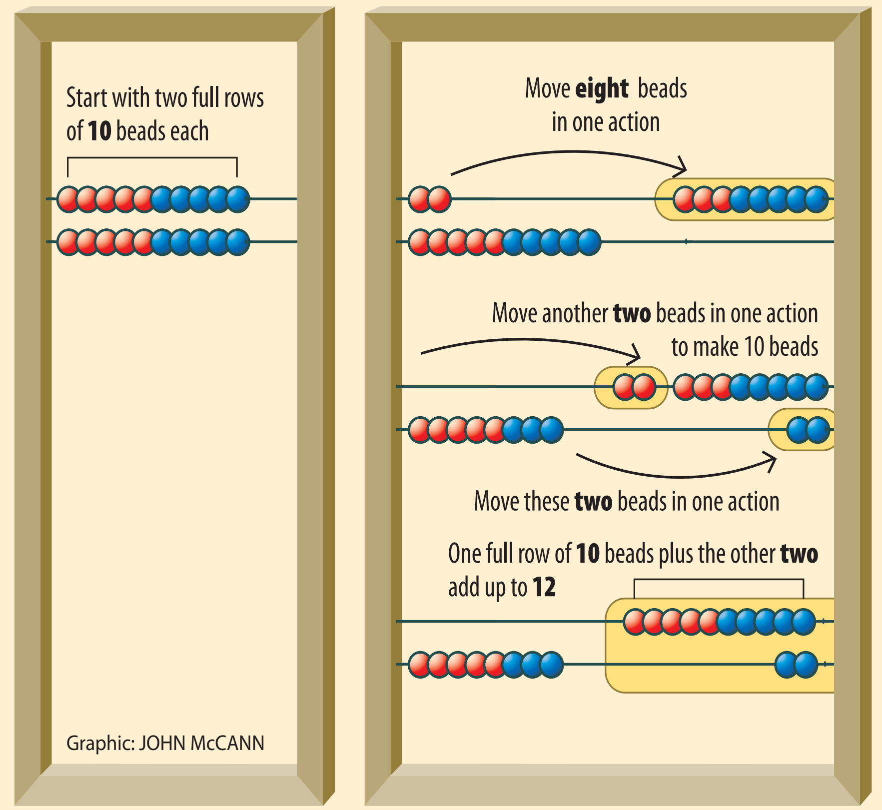
In the Foundation Phase, children often use counters or an abacus to work out sums like 8 + 4. If a child works this out by counting eight beads one by one on an abacus, and then counting four more beads one by one on the second row, and then counting them all up one by one to get 12, the teacher has several options on what to model as a more efficient way of working out the answer.

One option is to show that on an abacus we can "see" eight as a partition of 10 into eight and two and move a group of eight beads across in one action, rather than counting one by one.

The four that needs to be added on can then be moved in two actions – the remaining two on the same row, and two more on the next row. Then, rather than counting out again to get the total, we can see that the answer is shown as a full row (10) and two on the next row.

Using an abacus to work out sums

Breaking 10 beads into eights and twos to make 12



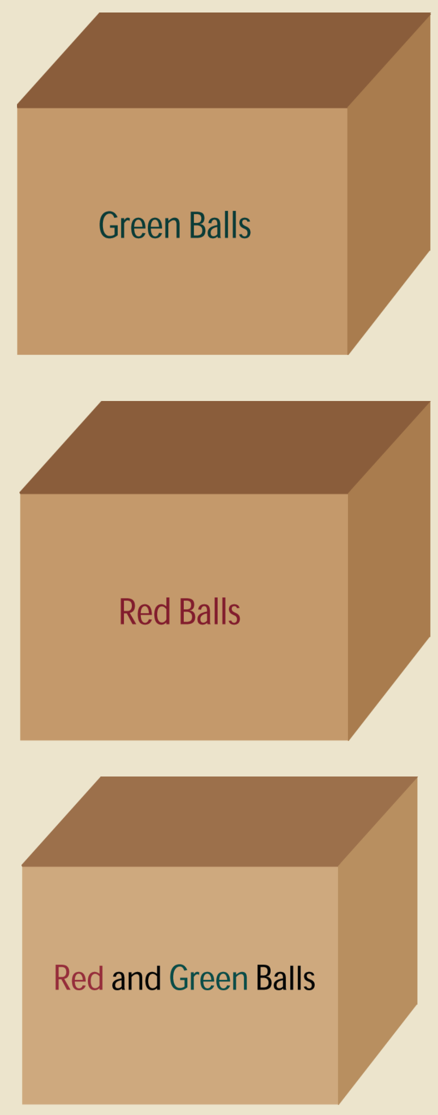
In abacus form (refer to graphic above), we can represent the steps and some of the teacher talk like this:

- eight moved with one action
- two moved with one action
- two moved with one action
- One full row (a 10)
- And a two makes 12

We believe that our job as teachers is not just to accept the methods that learners bring but to use these as a springboard for building higher level learning.

We also believe that being aware of more sophisticated strategies and being able to provide and explain models of these strategies in class is an important part of of mathematics teaching. Such work will be important if we are going to improve the poor performance in mathematics that we currently see in South Africa.

Wits Maths Circle Problem Set 6:



Question 1:

There are three closed boxes in front of you. One box contains only red balls, one box contains only green balls and one box contains a mixture of red and green balls. Each box has one of the following labels on it: green balls, red balls, and red and green balls.

However, each box is incorrectly labelled. You must choose one box and reach in (without looking inside) and grab one ball. After you have done this you must relabel the boxes correctly. Explain which box you must choose in order to be able to do this and why this allows you to relabel the boxes correctly.

Question 2:

You have three jugs. One holds eight litres, the second five and the third three. The eight-litre jug is full of water. You need to measure four litres. How can you pour liquid between the jugs to end up with exactly four litres in one of the jugs.

Rounding to the nearest 10

Recently we were working with some learners and asked them to round 237 to the nearest ten. We were surprised to find that quite a large proportion of the learners couldn't do it

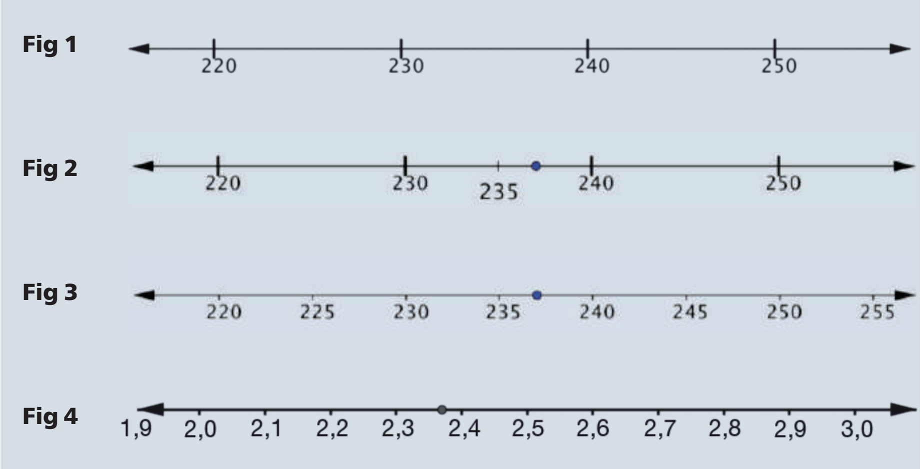
A little probing to try and find out what the problem was revealed that the core issue was that these learners didn't really understand what the question was asking them to do. This was typified by Lindi's response to our question "When we ask you to round 237 to the nearest ten, what do you think we are asking you to do?"

Lindi: "You're asking me whether the tens are bigger than 5….uh, no whether the units are bigger than 5 or something like that".

The fact that Lindi's recall of rounding had to do with some poorly remembered procedure rather than a core understanding of what she was trying to do when rounding, goes a long way to explaining why she struggled to produce the correct answer.

We found that some work with these learners using a number line was very useful in helping them build a clear understanding of what they were doing. Our work went something like this. First it was important to check that they knew what a "ten" is. So we all counted up in tens together and then produced a portion of the number line with the tens close to 237 (see Fig 1). We discussed the specific tens that surrounded 237. This narrowed our attention to 230 and 240. A little further exploration of this small section of the number line allowed us to note that, since 235 is the halfway mark between 230 and 240, 237 had to be a bit more than halfway between 230 and 240. The learners also saw that the closest ten to 236, 237, 238 or 239 would be 240 and the ten closest to 231, 232, 233 and 234 would be 230. And then we discussed the fact that, as 235 is equidistant from 230 and 240, mathematicians have simply adopted a convention which says that for numbers exactly halfway along we round up (see Fig 2).

The number line gives a very clear picture that what we're trying to do when we "round 237 to the nearest ten" is simply establishing the ten that is closest to 237. This meant that when we asked these same learners to "round 237 to the nearest five", they produced a number line marked with the fives near 237, placed 237 on this number line and saw that 235 is the nearest five (see Fig 3).



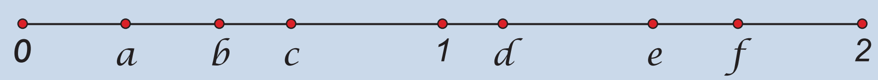
When we need to round decimals we can use the number line too. If we need to round 2,37 to the nearest tenth, we need a number line marked with tenths. In a similar way to what we did before, we can focus our attention on the tenths that surround 2,37 namely 2,3 and 2,4. Again, some further exploration of this little section of the number line allowed us to note that, 2,35 is the halfway mark between 2,3 and 2,4, and thus 2,37 is a bit more than halfway between 2,3 and 2,4. Thus the nearest tenth to 2,37 is 2,4 (see Fig 4).

The number line is a very powerful tool and thus a useful aid for working with numbers with understanding. Learners who have a strong picture of the number line in their heads can draw on it to solve novel problems that involve numbers. Consider, for example, your answer to the question: Round -1,7 to the nearest integer. A quick sketch of the number line, placing -1,7 on it and an understanding that we're looking for the integer nearest to it makes this problem easy to answer with -2.

Wits Maths Circle Problem Set 7:

1. This question comes from the 2013 AMESA Mathematics Challenge for grade 7s. You can find more lovely questions from their Challenges at *www.amesa.org.za*

On the number line below which value *(a; 1; d; e or f)* best represents $b \times c$. Explain why you say so.



2. Circle the number nearest in size to the answer (do not work out the arithmetic)

a) 3,1 × 6,9

0,002 0,02 0,2 2 20

b) 0,31 × 6,9

0,002 0,02 0,2 2 20

Learning times tables

The trick to remembering times tables lies in understanding them, as well as practising them

In our experience of working with grade eight and nine mathematics learners we have noticed that many of them don't know their times tables fluently. This really hampers their ability to engage with the new mathematics they are trying to learn.

Their teachers often ask: "Why are they not teaching children their times tables at primary school?"

We know that at most primary schools there is a lot of times tables practicing taking place, so what is going wrong?

This question was asked to Dr Ban Har Yeap, a mathematics education expert from Singapore, on his recent visit to South Africa.

His answer was as follows: "The problem is that we try to drill learners to memorise the times tables without doing the necessary work which allows them to understand multiplication, and reason about how multiplication facts are related."

Yeap does not deny that lots of practice with multiplication facts is necessary to become fluent in the times tables, but he argues that learners are far more likely to be successful in learning them if that practice is built on a meaningful understanding of connections between multiplication facts.

Below is a suggested series of activities for reasoning about multiplication facts using the example of the six times table.

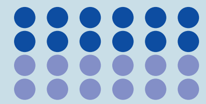
Visual images of multiplication

It is very helpful to have a mental image of multiplication as an array. For example, if children can see 2 × 6 as two rows of six circles, then it is easy to see that 2 × 6 is the same as 6 + 6.

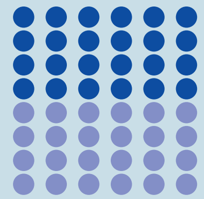
Using visual images along with known facts to derive new facts

Following some discussion of the visual image for 2 × 6, we can ask learners if they can figure out how to adapt this image to figure out what 4 × 6 is.

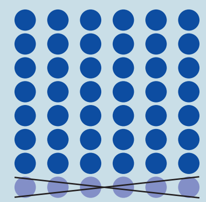
As teachers, we are listening for reasoning that if two rows of six are 12, then 4 × 6 is just two rows of six and another two rows of six, so it is double 12, which is 24.



And from there, 8 × 6 is easy to see as four rows of six, plus another four rows of six i.e. 24 + 24 = 48.



And from there learners might be able to reason that 7 × 6 is those eight rows of six, take away one row of six i.e. 48 – 6 = 42.



Using a clue board

Once learners have a strong visual image of multiplication and have played with using that image to figure out how to derive new multiplication facts from known ones, it is often helpful to progress to getting learners to draw up "clue" boards. They can then work with these boards in class discussions to reinforce how they can be used to derive all the facts related to the first 10 or 12 multiples in the 1-10 time table.

Learners can fill in the clue board for the six time table using doubling:

	× 6
1	6
2	
4	
8	
10	

Step 1: We know 1 × 6 is 6

Step 2: Since we know 1 × 6 = 6 we can double this to get 2 × 6 = 12

Step 3: Since we know 2 × 6 = 12 we can double this to get 4 × 6 = 24

Step 4: Since we know 4 × 6 = 24 we can double this to get 8 × 6 = 48

Step 5: We know 10 × 6 = 60

So the completed table looks like this:

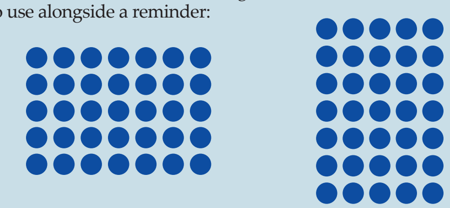
	× 6
1	6
2	12
4	24
8	48
10	60

Now with that clue board in front of them, ask learners how they might use the clue board to figure out what 7 × 6 is.

Give learners time to think and discuss, and encourage different ideas from the class. Some might use the idea of saying 48 – 6 = 42 since if you have eight lots of six, and take away one lot of six you'll get even lots of six. Others might say 6 + 12 + 24 = 42, since one lot of six plus two lots of six plus four lots of six gives seven lots of six. Similarly, 9 × 6 can easily be derived using 60 – 6 (10 × 6 – 1 × 6) or 48 + 6 (8 × 6 + 1 × 6).

Lots of practice and pattern spotting

And then there is just a need for practice; but that practice will yield better results if the reasoning part is constantly reinforced. Array images can be turned through a quarter turn to show that 7 × 5 and 5 × 7 give the same answer. This reasoning can be helpful to use alongside a reminder:



Or a learner who is stumped by 12 × 9 might need to be asked about how to construct 12 groups of 9 as (10 × 9) + (2 × 9), and that this provides a way of figuring the answer out quickly. Similarly, noticing that the six times table is double the three times table makes the six times table easier to learn.

Lots of practice does help children to learn their times tables "off by heart", but if it is done together with reasoning about multiplication and making connections between the different facts it will be more effective and help to ensure that learners have efficient strategies for figuring out the multiplication results quickly — even if they have forgotten individual facts.

Wits Maths Circle Problem Set 8:

The problem below is from the NRICH website — a really good source of interesting maths problems. *http://nrich.maths.org/730*

In the multiplication below, some of the digits have been replaced by letters and others by asterisks. Where a digit has been replaced by a letter, the same letter represents that same digit and different letters represent different digits. (e.g. if A = 1 in the first line, then A = 1 in the second line. And A and B must be different digits). Asterisks can stand for any digit and can be the same or different, and can also be the same as A, B or C. Can you reconstruct the original multiplication?

```
      A B C
      B A C
    -------
    * * * *
    * * A
* * * B
-----------
* * * * * *
```

Making sense of problems

Using diagrams to teach and solve word problems

We work with students who are training to be primary school mathematics teachers. In the courses we teach, we try to make sure that student teachers develop a very deep conceptual understanding of the maths they are going to teach.

We have found that using story sums and diagrams are a very useful way to challenge and develop the student teachers' (and our own) understanding of the mathematical ideas they are working with. At the moment we are working on fractions and have found that getting to grips with them in a deep way is not at all simple.

We thought you might find it interesting to work through some of the tasks we give our students and see whether they challenge you or give you something to think about. We have found that moving between the words in story sums and diagrammatic representations of key features of the story provides us with windows into seeing and assessing the student teachers' understanding of problems.

We have also found that encouraging student teachers to draw diagrams representing story situations helps them to develop their ability to make sense of the problem in ways that help them to decide what number sentence is needed to solve the problem, rather than trying to guess and simply hoping for the best.

We sometimes give our students a number sentence involving fractions and ask them to create a story that can go with it. Part of what we are trying to do here is to build connections between problems and situations, but we also find this way of working useful for breaking the expectation that number sentences are simply cues for unthinking operations

We would strongly encourage primary maths teachers to share story situations and diagrams connected to these situations with their pupils as well. Discussions about which diagrams are correct, and which are helpful to answering problems, are likely to help to overcome the widespread complaint that pupils "cannot do word problems".

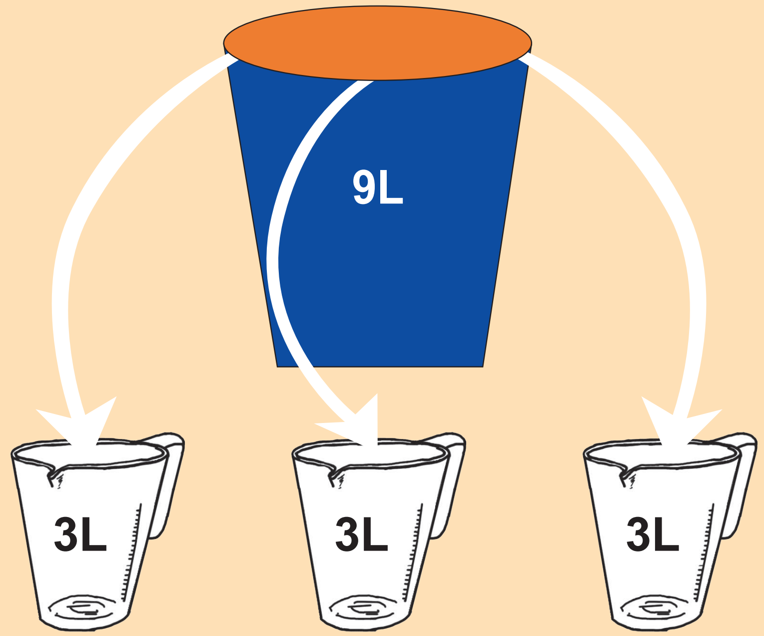
A common response is either to avoid word problems or to present them with the procedure needed to get the answer. Either way, pupils do not gain experience in having to figure out how to represent the situation, and how to go about selecting the procedure needed for finding the answer.

We all get good at the things that we practise. Practising working with story situations and representing situations with diagrams is, we believe, time well spent for the learning of maths.

Fraction stories and diagrams

I squeeze oranges and end up with 9 litres of orange juice.

A If I pack the juice into 3 litre jugs, how many jugs would I need?



Number sentence for this situation is $9 \div 3 = 3$

B Think about how this diagram would have to be edited if I decided to pack the juice into $^1/_2$ litre jugs instead of 3 litre jugs.

And what number sentence matches this situation?

Wits Maths Circle Problem set 9:

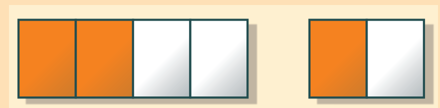
Using diagrams to understand the structure of story situations

1

If the rectangle below represents a sheet of paper that is ¾ of a larger sheet of paper, what would the larger piece of paper look like?

2

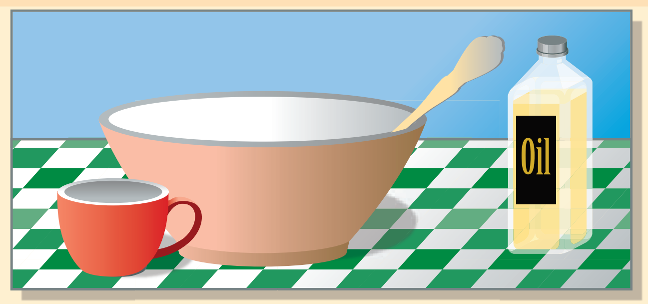
A learner says that $^2/_4$ is clearly bigger than ½ and produces the following diagram to convince you:



What would you say to the learner?

3

I am making a recipe that calls for $^1/_6$ of a cup of oil. I have a bottle that contains $^2/_3$ of a cup of oil. What fraction of the oil in the bottle should I use for the recipe? Draw a diagram to help you solve the problem and explain your answer.



4

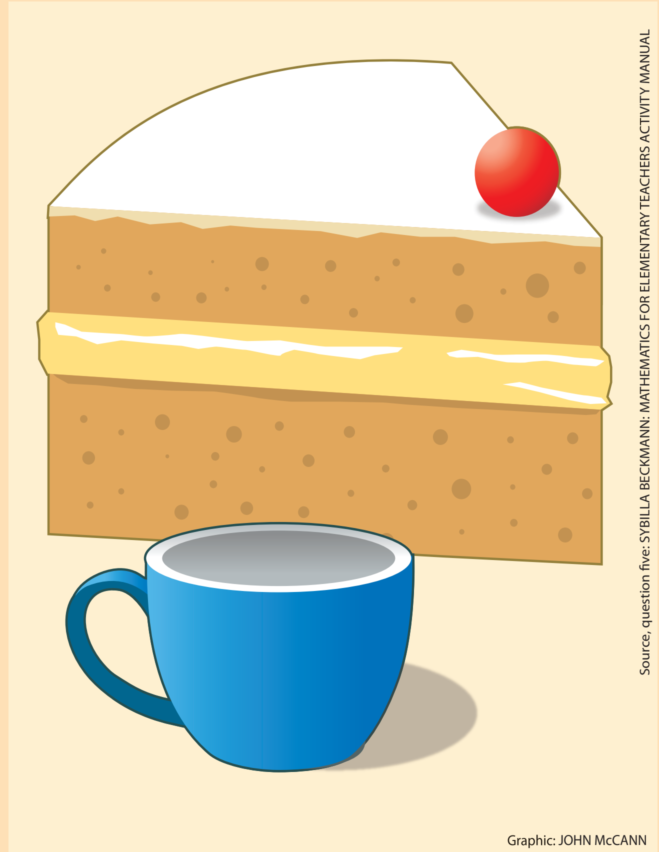
Make up a story sum that would require you to do the calculation $^2/_3 - ^1/_2$

5

Which of the following problems are story problems for $^2/_3 \times ^1/_4$ and which are not? Why?

a) Joe is making $^2/_3$ of a recipe. The full recipe calls for $^1/_4$ cup of water. How much water should Joe use?

b) There is $^1/_4$ of a cake left but $^2/_3$ of Mrs Watson's class would like to have some cake. What fraction of the cake does each student who wants cake get?



Source, question five: SYBILLA BECKMANN: MATHEMATICS FOR ELEMENTARY TEACHERS ACTIVITY MANUAL

Graphic: JOHN McCANN

Thinking about teaching fractions

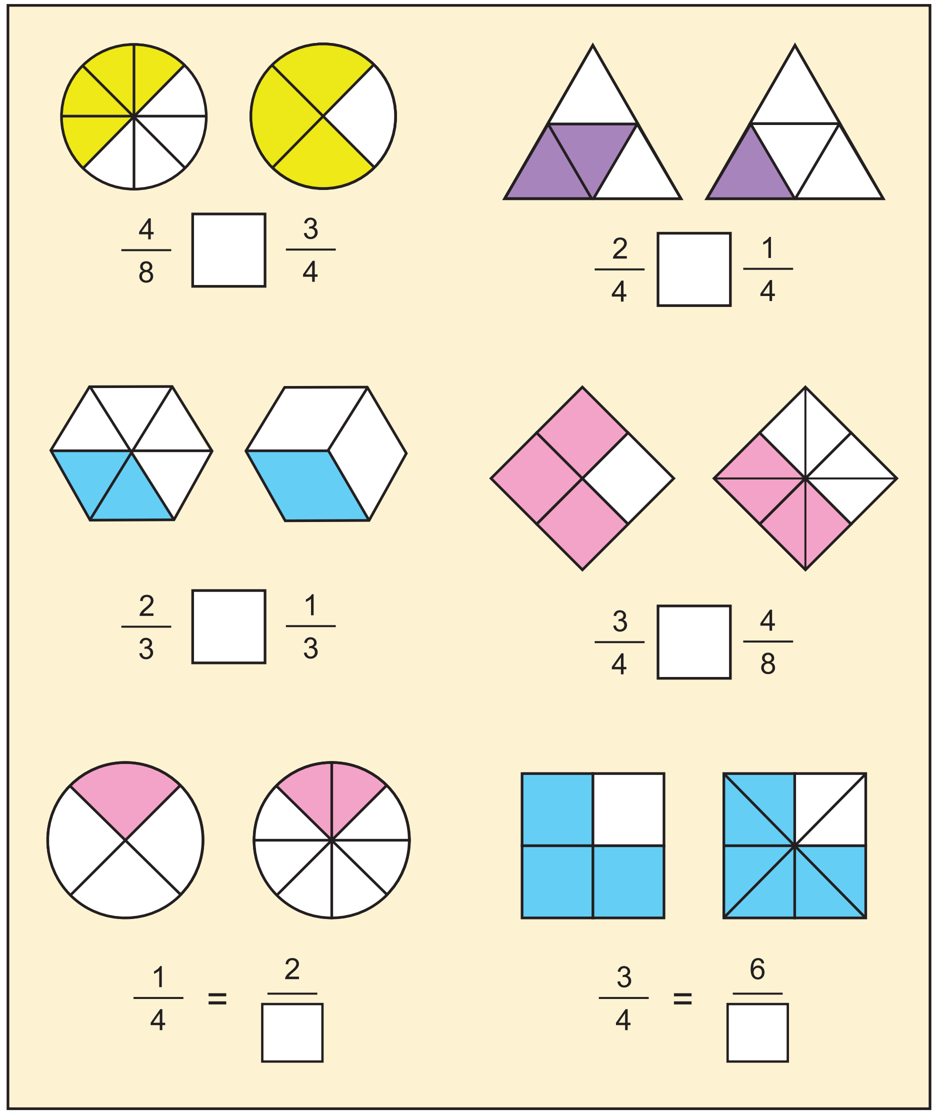
The old age states' Those who can, do. Those who can't teach'. The reality, as anyone who has spent time in primary classrooms, is a great deal more complex than that. So what does it take to teach primary mathematics? At the most basic level, the answer to this question is obvious. The teacher must know the mathematics they have to teach. Yes, but evidence gathered across a range of countries shows that 'knowing the mathematics' - as attested to by exit level school qualifications- often involves knowing 'how' to do without an appreciation of why we do things in specific ways. Examples from primary maths help to illuminate this issue.

Most people will recall learning at some point or other in their schooling that when dividing fractions like $^{2}/_{3} \div {}^{1}/_{5}$ the rule to use is 'invert and multiply' or 'tip and times' Essentially, the rule tells us to turn the second fraction upside down and to change the division sign into multiplication sign i.e. $^{2}/_{3} \times {}^{5}/_{1}$. We then have to remember how to multiply fractions - multiply tops, multiply bottoms, simplify- and the question is answered. Most of us can probably perform this sort of calculation, but most of us probably can't explain why it works or what it means.

But research shows that learning maths in this way, as layers of rules upon rules, is not very efficient even in the short term, and becomes increasingly problematic as the number of rules to memorise mounts up as children progress through schools.

So how might teachers need to understand problems such as these in order to work more productively in classrooms. What kind of things might a teacher need to know to be able to make meaning of problems like division by a fraction?

Think for a moment about what it would take to teach children how to do $3 \div {}^{1}/_{2}$? Why does it make sense that the answer becomes $3 \times {}^{2}/_{1} = 6$? Let us look at the kind of knowledge a teacher would need to have in order to do this:



- The teacher needs to know how this problem connects to prior work the children in her class have seen. S/he needs to understand how this problem requires extensions or shifts in their ways of understanding more familiar problems. In this case, the teacher might see that the fractions problem involves division, and know that the class will need to have encountered simpler, whole number division problems previously. So for a child who asks, or does not have an answer to what $3 \div {}^{1}/_{2}$ means, a follow up question might be ' So what does $15 \div 5$ mean?' Writers in mathematics educations have pointed out that children frequently answer questions like this in terms 'of 'It means that 15 is shared into 5 groups'. This is a correct answer, but it is not a conception of division that extends well to fractions as it does not intuitively make sense to divide a quantity into ½ groups. The teacher in this situation needs to know of alternative ways of thinking about that division that might lead better into the case of division involving fractions. Another way of thinking about divisions is to say that $15 \div 5$ can be considered in terms of 'How many 5s are there in 15?' If this is the case for $15 \div 5$, then we can think about our original problem in terms of 'How many ½'s are there in 3?
- The teacher needs to have different ways of presenting the problem so all children can see the solutions. A diagram can help a class to see that if we try to cut a ribbon that is 3 meters long into pieces that are ½ meter long, we get 6 pieces of ribbon. The teacher could then show them that the magical 'invert and multiply 'rule produces this answer. The rule is useful, but using it with a sense of what the questions is asking for, and with a sense of being able to see roughly what the answer should be through estimating, is a much more powerful way of thinking mathematically than a blind application of the rule.

These ideas can be extended to questions like $1^{3}/_{4} \div {}^{3}/_{4}$ and $^{1}/_{2} \div {}^{1}/_{3}$.

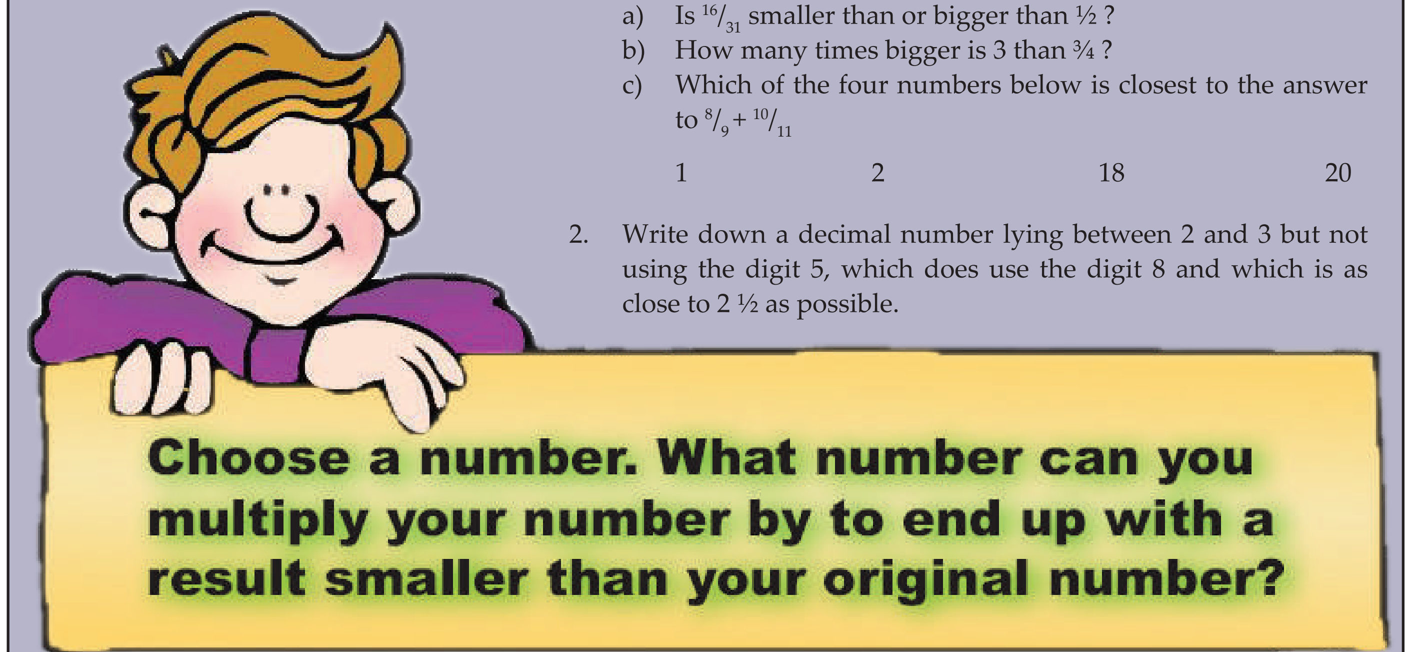
In mathematics classrooms, we see instructions on 'What to do and 'how to do 'more often than we see 'why we do' questions and the kinds of explanation described above. This extended knowledge base forms the underpinning of 'how to teach' mathematics. Pre-and in-service teachers need access to this knowledge base if we are going to begin to address the problems of poor performance in our education system - problems that are exacerbated by teaching that emphasizes a plethora of rules for solving problems in numeracy and mathematics, whilst paying little, if any attention, to making sense of what these problems are actually about.

Wits Maths Circle problem set 10:

1. Provide answers to the following questions without doing any lengthy calculations. Explain your reasoning
   a) Is $^{16}/_{31}$ smaller than or bigger than ½ ?
   b) How many times bigger is 3 than ¾ ?
   c) Which of the four numbers below is closest to the answer to $^{8}/_{9} + {}^{10}/_{11}$

      1 2 18 20

2. Write down a decimal number lying between 2 and 3 but not using the digit 5, which does use the digit 8 and which is as close to 2 ½ as possible.



A fraction tea party

A good under standing of fractions is crucial

Fractions are known to be difficult to both teach and learn. However a good understanding of fractions is crucial to the development of proportional reasoning and provides the basis of further study in mathematics.

The difficulties we face in the classroom when teaching fractions make it tempting to simply present learners with a set of clear rules that they have to follow — for example in dividing fractions you "tip and times", in multiplying fractions you multiply numerators and multiply denominators but in adding fractions you don't add numerators and add denominators etc. But unless the rules are rooted in a firm understanding they become meaningless, easily forgotten or confusing.

They can't provide the necessary conceptual development required for further mathematics studies. In this article we share with you one activity that can be useful in building an understanding of fractions. The activity is the fraction tea party (See the diagram alongside) .

The last two scenarios require a little more thought. If we have fewer friends as well as fewer brownies, it is tempting to say each person will get the same amount of brownie as they did on Monday. But the problem is we don't know the actual change in the number of brownies and the number of people. So, for example, if I had 10 friends and 10 brownies on Monday they'd each get one.

If on Tuesday I had fewer friends and fewer brownies it is possible that I had five friends and nine brownies (in which case each person would get more) or it is possible that I had eight friends and two brownies (in which case each person would get less) or it is possible that I had five friends and five brownies (in which case each person would get the same). So in the case where there are fewer brownies and fewer friends we can't say whether each person will get more, less or the same as the friends on Monday did.

The fraction tea party

A classroom activity

Whenever I have friends round for a tea party I insist they share the brownies equally. On Monday I have a tea party — I invite a group of friends and bring some brownies for them to share equally.
On Tuesday I have another tea party and invite more friends than I had on Monday, but have the same amount of brownies as I did on Monday. How does the amount of brownies each friend gets on Tuesday compare to Monday?

On Wednesday I have another tea party. I invite the same number of friends as I did on Monday, but buy more brownies than I did on Monday. How does the amount of brownies each friend gets on Wednesday compare to Monday?

Complete the following table by filling in the words *more*, *less*, *the same amount* or *can't tell* to compare how much each guest will get with the amount of brownies each guest got on Monday.

Compared to Monday	Fewer brownies	Same number of brownies	More brownies
Fewer friends			
Same number of friends			
More friends			

Completing the table can be done by using everyday reasoning. It is obvious to me that if I have the same number of brownies, but fewer friends, then everyone is going to get more. If I have more brownies and fewer friends then they'll also get more. This allows me to complete the table as far as this:

	Fewer brownies	Same number of brownies	More brownies
Fewer friends		More	More
Same number of friends	Less	Same	More
More friends	Less	Less	

How does this relate to fractions?
Notice that in the tea party problem, we used a sharing or "division" scenario involving "number of brownies ÷ number of friends". As a teacher, I can use this task to make a connection between:

$$\text{Number of brownies} \div \text{number of friends} = \frac{\text{number of brownies}}{\text{number of friends}}$$

Graphic: JOHN McCANN

We can make a similar argument for the situation involving more brownies and more people.

In this way, the teacher relates division notation to fraction notation — a connection that many learners are

unaware of. This lack of awareness is not surprising if the only way in which teachers explain fractions is in terms of "parts of a whole'' in the context of pizzas or cakes.

Thinking about sharing brownies between friends allowed us to fill in seven of the nine boxes in the table without knowing the exact numbers of brownies and friends. The absence of specific numbers is useful for communicating the idea that fractions are centrally about a relation between two numbers — the numerator and the denominator. Specific numbers tend to push teachers and learners towards "calculating the answer'', without stopping to understand the idea.

The idea was enough to fill in most of the boxes. But when the "fewer brownies/ fewer friends'' or "more brownies/ more friends'' scenarios came up, we did think about options for the actual numbers to decide that the amount of brownies that each person got could be the same, or more, or less – depending on the actual numbers.

Here, we are starting to compare actual fractions. So we can compare, for example, five brownies shared between four friends with six brownies shared between five friends. (Diagram above.)

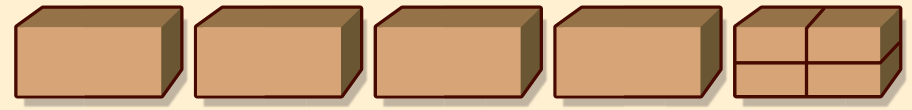
There are other possible ways to picture the sharing that can inspire rich conversations about fractions. We can also think about how we can use the diagrams to compare 5/4 and 6/5. We can go on to compare scenarios like "seven brownies, nine friends on Monday'' compared with "eight brownies, 11 friends on Tuesday''.

Again we can use diagrams at this point to see if we can work out which fraction is bigger. We get to a point where "lowest common denominators'' come in, not as a rule – but as a well understood and useful way of comparing fractions.

Tea party solutions

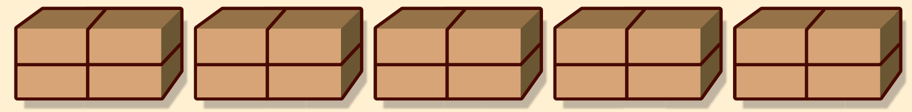
Finding the answer

To figure out how much each friend gets in the various scenarios it is useful to draw pictures. The fact that sharing is a very common everyday occurrence makes it possible for learners to think up different ways of reaching the answer. For example, one could picture sharing five brownies between four people as follows:



Each person gets one brownie. Then the last brownie is cut into four portions and each person gets a piece. So we can see that each person gets $\mathbf{1\frac{1}{4}}$ brownies. Thus $\mathbf{5 \div 4 = \frac{5}{4} = 1\frac{1}{4}}$

Alternatively, you could picture sharing five brownies between four people by cutting each brownie into four portions and giving each person a piece of each brownie.



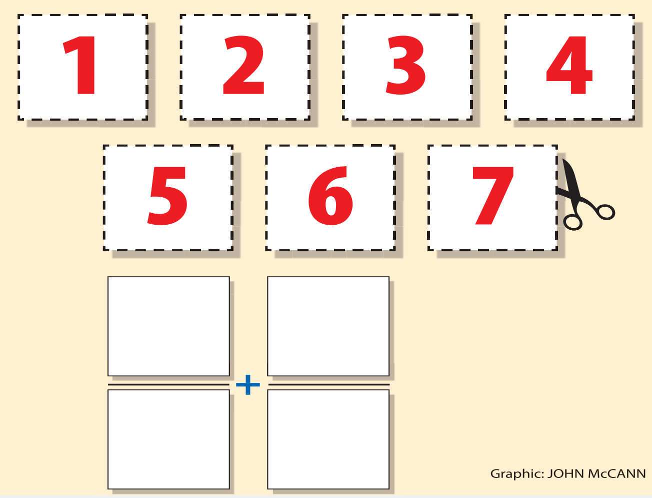
This allows us to see $\mathbf{5 \div 4}$ as five pieces of portion size $\mathbf{\frac{1}{4}}$

Graphic: JOHN McCANN

Wits Maths Circle problem set 11:

Adding fractions

Cut out the numbers below. Choose four numbers that you can arrange in the fraction template below to make the sum as close to 1 (but not equal to 1) as possible.



Graphic: JOHN McCANN

What to include when teaching fractions?

We have thought about and written a lot about ideas related to teaching and learning fractions. We revisit some of those ideas here in a broader discussion about 'what' we should be focusing on within this teaching.

Consider these two problems:

$^4/_5 + ^1/_3$ equals ☐

How many fractions lie between ¼ and ½?

Our work with upper Intermediate Phase and Senior Phase Learners suggests that more of them can answer the first questions than the second question. When we ask teachers whether they think children should be able to answer both of these questions, they usually say yes. But when we ask them what teaching they have done to support this learning, they say that while they have taught children the procedure for working out the answer to the first question, they have not done any teaching that supports children to answer the second question.

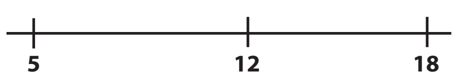
We think that the second question is important for communicating an important idea about the nature of the kinds of numbers that we deal with in primary school. This ideas begins with tasks asking:

'Give me any two numbers between 0 and 20.' E.g. 5 and 18.

Mark these two numbers on a number line:

To another learner: 'Give me a number that lies between these two numbers.'

E.g. 12.[Mark 12 on the line]:



Wits Maths Circle problem set 12:

Questions

To celebrate World Maths Day a school asks each of the learners from the senior grades to partner with a learner from the junior grades so they can work together as a team on a maths quiz on the World Maths Day. So far $^2/_3$ of the learners from the senior grades have partnered with $^3/_5$ of the learners from the junior grades. What proportion of learners at the school has got partners for world Maths Day?

Ask for a number between 5 and 12 . And continue to ask for numbers between the lower number and the number offered. Eventually, we are likely to get to asking for a number between 5 and 6.

Children in Foundation Phase can usually offer ' 5 and a half' here, which can also be marked in. In Intermediate Phase, we continue this game and end up at questions like: 'Give me a number between 5 and 5 a quarter'. At this point some children tell us that there are 'no more numbers in between'. We push with questions asking about whether there is a 'number half way along'. Children who have seen fractions wall diagrams will say that there is an eighth halfway along, making this number five and one eighth.

We push again, for another number in between 5 and 5 and an eighth.

The idea we are trying to develop is that we can ALWAYS find a number between two given numbers, regardless of how small the two numbers are and whether they are fractional numbers. And if we continue this idea, it must mean that there are an infinite numbers of numbers between any two numbers. This idea of the density of numbers on the numbers line is an important idea in mathematics. It helps build the numbers sense that we have frequently talked about across our articles, and questions like these help children to visualize the positions and relative size of the numbers as well, thus supporting a much broader set of skills related to number and operations.

We are not asking teachers NOT to teach the procedures associated with adding and subtracting fractions. But we are saying that teaching these procedures without giving children a sense of what fractions are as numbers and how they fit in to the broader number system is not very helpful.

Decimals and remainders

We posed the two questions below to our student teachers and discovered that the answers were not obvious to everyone in the class. In fact, the answers were quite difficult to explain clearly.

1 We know that $^{20}/_8 = ^{10}/_4$ and we know that we can interpret these fractions as division so that $20 \div 8 = 10 \div 4$. But if we calculate $20 \div 8$ we get *2 remainder 4* and if we calculate $10 \div 4$ we get *2 remainder 2.* **Why do we get different answers?**

1 I divided a number by five on my calculator and the answer was 34.2. **If I had to write the answer using the form 34 *remander R*, what would the remainder R be?**

Both of these questions rely on understanding the meaning of division, decimals, fractions and remainders.

Let's look at the first one question.

When we say $^{20}/_8$ we can interpret it as 20÷8. One way we can think of 20÷8 is as follows: If I have 20 smarties and divide them into packets with 8 smarties in each packet, how many packets will I have?

The picture of each packet will look like this:

Four smarties is half a packet of Smarties because we have 4 smarties out of the 8 we need to have a full packet i.e. $^4/_8$=½. So, what we have is 2 packets remainder 4 smarties or 2 ½ packets or 2.5 packets.

When we say $^{10}/_4$ we can interpret it as 10÷4.

One way we can think of 10÷4 is as follows: if I have 10 smarties and divide them into packets with 4 smarties in each packet, how many packets will I have?

The picture of each packet will look like this:

Two smarties is half a packet of smarties because we have 2 smarties out of the 4 we need to have a full packet i.e. $^2/_4$=½. So what we have is 2 packets remainder 2 smarties or 2 ½ packets or 2.5 packets.

The second question requires using the understanding we got from the first question.

We know that 34.2 means 34 $^2/_{10}$ = 34 $^1/_5$.

We can think of dividing by 5 as putting smarties into packets of 5. The answer tells us that we made 34 full packets and $^1/_5$ of a packet.

If each packet contains 5 smarties, then the $^1/_5$ of a packet is just 1 smartie — so the leftover or remainder is 1.

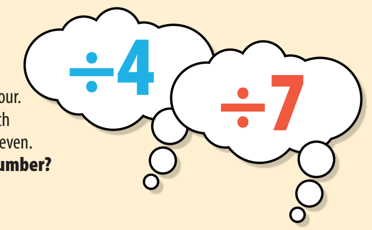
Graphics: JOHN McCANN Maths source: BOWIE, VENKAT

Wits Maths Circle Problem set 13:

Working out the remainder, an amount left over after dividing numbers

Question one

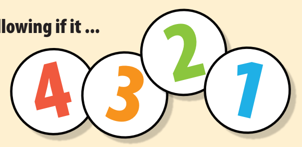
Lerato is thinking of a whole number, which leaves a remainder of one when divided by four.
Bongani is thinking of a whole number, which leaves a remainder of one when divided by seven.
Could they be thinking of the same number?



Question two

Can you find one number that does all of the following if it ...

- leaves a remainder of four when divided by five?
- leaves a remainder of three when divided by four?
- leaves a remainder of two when divided by three?
- leaves a remainder of one when divided by two?



Answers on page 14

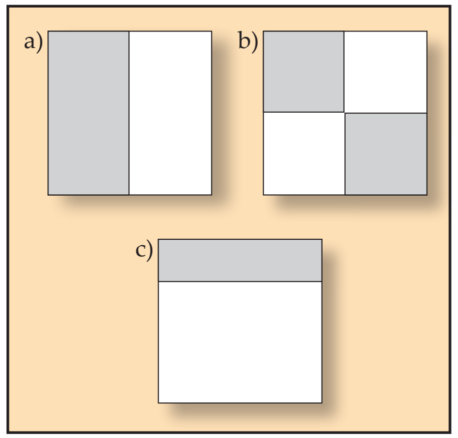
Graphic: JOHN McCANN Quiz source: NRICH

Picturing $^1/_2$

The National Council of the Teachers of Mathematics (NCTM) is the organization for mathematics teachers in the USA and has a website *(www.nctm.org)* which contains all sorts of lovely resources for mathematics teachers. We've currently been reading a book we bought on their website called '' Putting Essential Understanding of Fractions into Practice'' and wanted to share some ideas triggered by a chapter in this book.

One of the reasons we provide tasks to our learners is for us, as teachers, to get feedback on their understanding and to identify any misconceptions they have. In the NCTM book they discuss some grade 3- 5 learners' responses to the following task:

Do the brownies below Have ½ a brownie shaded? Explain your thinking.



Learners' responses to the questions differed, but some learners' responses revealed some core misconceptions. Most learners were happy to say that (a) does represent ½ because the brownie is split into two pieces. This might lead us to assume that they've understood the concept. However many of those same learners would make a statement like the following for (b): "No, because half is 2 pieces and this is 4 pieces'' revealing a misconception that ½ means that the object must be cut into EXACTLY two pieces. Some learners might extend this thinking to say (c) does represent ½ because there are two pieces, but most will recognize that the pieces are not equal and so doesn't represent ½.

However it is interesting to watch the language learners use when they discuss (c). We've seen a number of learners say something like ''NO because one half is bigger than the other half'' which again suggests thinking that equates cutting into two pieces with one half.

For learners, seeing ½ as a single quantity that can be represented in a number of different ways rather than being tied to one representation (that of being cut into two pieces) takes time and multiple different experiences. Interestingly some of the learners who stated that (b) did not represent ½ when asked, in a later question, to give fractions equivalent to ½ happily provided the following list:

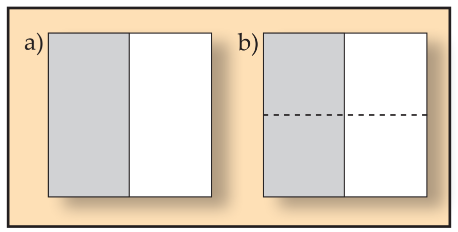
$$\frac{2}{4};\frac{3}{6};\frac{4}{8};\frac{5}{10}$$

In other words, knowing the procedure for finding equivalent fractions didn't necessarily mean that they understood what equivalent fractions are.

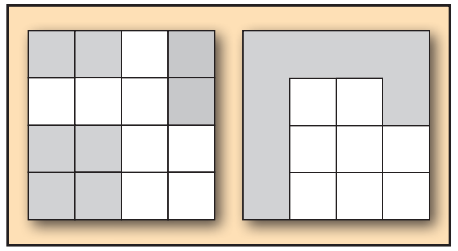
On the other hand, even learners who got most of the questions about diagrams that represented ½ a brownie wrong, were able to correctly identify which of those same diagrams represented a fair sharing of the brownie between two people. This understanding could be used to help challenge the misconception about representations of ½.

In addition to building on the fair sharing notion, questions that confront the misconception head on are important.

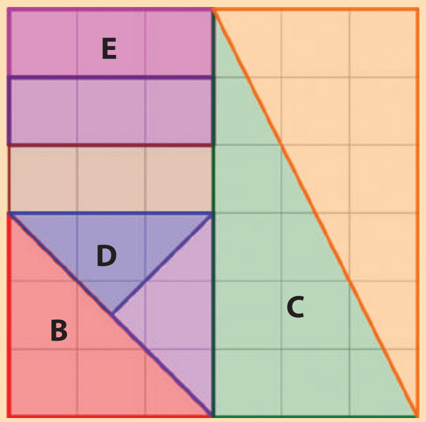
Compare the brownies on the left and right. Does the amount of brownies shaded change if I make and extra cut?



And even after tackling some of those misconceptions, it turns out that for some learners the fact that the shaded pieces don't touch (like in the pictures on the left below) or that the shaded and unshaded pieces are totally different shapes (like in the picture on the right below) makes it hard for them to believe the shaded portions represent ½. So we need to make sure learners have experience of a range representations like these to support their building up to images of ½.



Wits Maths Circle problem set 14:



I have taken a square and cut it into pieces as shown.
What fraction of the square do we get when we put

a) The red triangle and the blue triangle together
b) The green triangle and the blue triangle together
c) The red triangle and the green triangle together
d) The green triangle and the pink rectangle together

Proportional reasoning

Visualisation can help pupils grasp this difficult but essential maths technique

Before reading further, look at the question illustrated on the right 'Mr Short and Mr Tall'. What answer did you get? What answer do you think pupils might give?

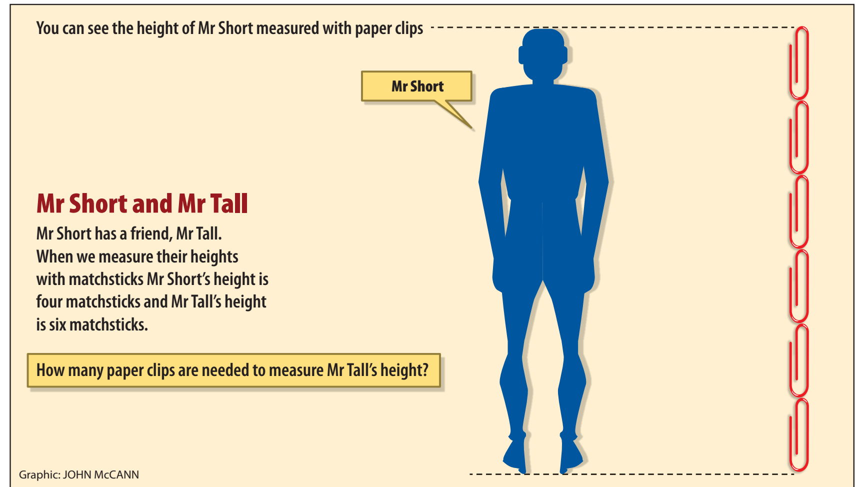
This problem has been used in research on proportional reasoning since the work of teachers Elizabeth and Robert Karplus and Warren Wollman in 1974. Quickly sketching the matchsticks next to the paper clips in the picture of Mr Short shows that each matchstick is 1.5 paper clips long. This means that Mr Tall's height will be nine matchsticks. However, this problem has been given to pupils (and teachers) in various countries at different levels of education and in most instances, they give the answer as eight matchsticks. When asked how they settled on eight, pupils often say that because there is a difference of two (in terms of matchsticks) for Mr Short and Mr Tall's height, they have "added the same amount" to the number of paper clips: 6 + 2 = 8 .

Now look at the following two questions. How do you think learners might respond to these? We are painting the school hall grey. I buy three tins of black paint and mix it with six tins of white paint and get the perfect grey colour.

1) When we run out of paint, one of the parents donates six tins of black paint. How many tins of white paint do I need to make the same colour grey?
2) When we run out of paint, one of the parents donates seven tins of black paint. How many tins of white paint do I need to make the same colour grey?

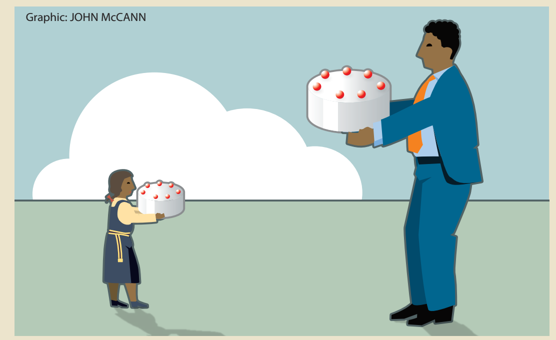
What research says

Interestingly, research suggests that what seems to happen here is that the first paint question with the "nice" numbers means that most pupils will answer correctly that 12 tins of white paint are needed. But then, in the case of the second question, pupils will resort to additive thinking and give the answer as 10 tins of white paint or they will reason that since seven is double three plus one, the number of tins of white paint should be 13 (which is double six plus one). Both these strategies are incorrect. The answer should be 14 because the numbers of tins of white paint must double the number of tins of black paint to get the same colour grey.



Graphic: JOHN McCANN

Using stories and images


Graphic: JOHN McCANN

Developing proportional reasoning is not an easy task. It is essential, however, because it is the foundation required for a sound understanding of fractions, decimals and percentages.

Unfortunately, because this type of reasoning is not easy and takes time and effort to develop, it is tempting to give up and simply tell pupils to use a "recipe" or "rule". We see many pupils getting ready to complete high school who know that: "If it is a ratio or proportion question I must cross-multiply", but they have little or no understanding of why and they pay no attention to the nature of the proportional relationships between the quantities involved.

Clearly, this type of reasoning needs to be developed over time. One way to help pupils is to encourage them to build images. Consider, for example, these class activities adapted from the work of a Dutch mathematics educator, Leen Streefland, for children of about eight years old.

First, you meet a friendly giant and decide to walk together to the shop. For each three steps the giant takes, you must take 15. If it takes the giant 18 steps to reach the shop, how many steps will it take you?

This is a scenario that can be acted out by the children and then modelled on a number line.

Second, you decide to buy ingredients to make a cake for your family. Your recipe tells you that you need:

2 cups of flour,
3 eggs,
1 cup of sugar and
1 tablespoon of butter.

The giant decides to make the same cake for his family, but will have to make it much bigger. If he buys 12 eggs for his cake, how much flour, sugar and butter will he need?

Both of you decide the cake would be nice if it had chocolate on it. You each buy the same size bar of chocolate that can be broken into 20 pieces, which you use to decorate the cake. Whose cake will have more chocolate?

These situations can be made more real for children by drawing diagrams to represent the ingredients in the first problem and the chocolate on the cake in the second problem.

Research strongly suggests that having these kinds of visual images associated with a range of problems provides a basis for building proportional reasoning in later grades.

Wits Maths Circle Problem set 15:

This problem is taken from a textbook from 1905 that the Mathematical Association of America put up on its website. It relates to proportional reasoning and we thought it would be fun for you to try.
If 40 oranges are worth 60 apples, and 75 apples are worth seven dozen peaches and 100 peaches are worth one box of grapes and three boxes of grapes are worth 40 pounds of pecans, how many pounds of pecans can be bought for 100 oranges? Source: Soulé's Philosophic Practical Mathematics (1905)

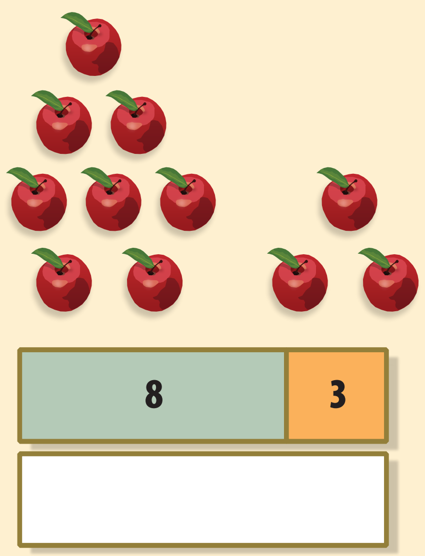
Thinking about change

Changing situations

Graphics: JOHN McCANN

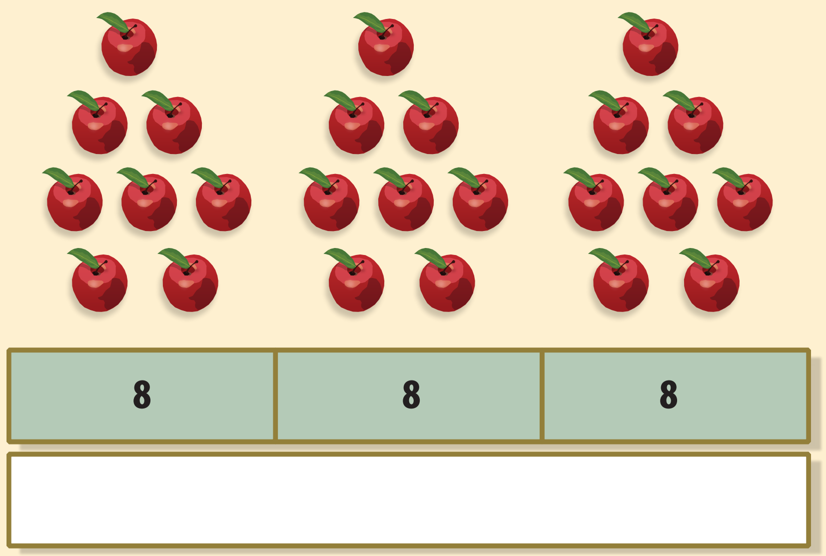
Visualising calculations and quantities

Lynn buys eight apples.
Hamsa buys three more apples than Lynn.
How many apples does Hamsa buy?



Write in the number of apples bought by Hamsa

Lynn buys eight apples.
Hamsa buys three times more apples than Lynn.
How many apples does Hamsa buy?



Write in the number of apples bought by Hamsa

In both situations, Hamsa buys a different quantity of apples to Lynn. This means that Lynn's quantity has to be changed in some way. Our interest is in HOW it changes.

In the first situation, the change is an additive one: 'Hamsa buys 3 more apples than Lynn'.

So we can work out what Hamsa buys with this sum:

8 + 3 = 11

In the second situation, the change is a multiplicative one: 'Hamsa buys 3 times more apples than Lynn'. So we can work out what Hamsa buys with this sum:

8 × 3 = 24

Children in schools often misrecognize multiplicative situations and deal with them by adding.

A common example is a problem like this one: 'A recipe says that 4 eggs are needed to make 1 batch of brownies. How many eggs would be needed to make 3 batches of brownies?'

Here, given that we are making 3 times as many brownies, we need 3 times as many eggs and so the answer would be 12 eggs. However, children often read the situation as 'We need 2 more batches of brownies, so we need 2 more eggs' and they write 6 eggs as the answer. When we give primary teachers (those training and those teaching) this question, they usually give the correct answer.

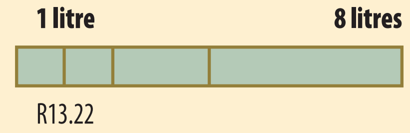
However, when we ask them how they got to that answer, they often show us a method they (and we) were taught at school: set the numbers up as two fractions and 'cross multiply'.

When we ask the teachers how they know what number to put where in the fractions they create, some of them say they are not sure — they just hope what they have done is correct. When we ask them why they cross multiply in this way, very few of them are able to provide a rationale.

There are better ways of communicating the reasons for the operations that we select. This contrasts with the 'rules without reasons' that dominate the methods above. For example, one option is to draw a diagram that represents the two situations: More generally, double number bars are useful for representing multiplicative or ratio based situations. Given the number and variety of situations, in everyday life and in mathematics, it is useful for teachers to be aware of them.

Here is one example:

If 1 litre of petrol costs R 13.22, how much would 8 litres cost?



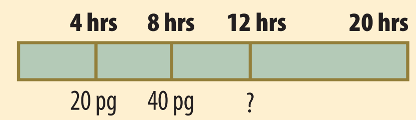
Write in the cost of two litres of petrol

Write in the cost of four litres of petrol

Write in the cost of eight litres of petrol

Ratio/rate situations can be represented with a double number bar because they involve relationships between quantities. Diagrams like this can help children to visualise their way into the calculation needed to work out the answer. We could, for example, work out how much 2 litres, then 4 litres and then 8 litres would cost here. And we can use this approach to help us answer questions like: If it takes me 4 hours to read 20 pages, how many pages will I read in 12 hours? Or in 20 hours?

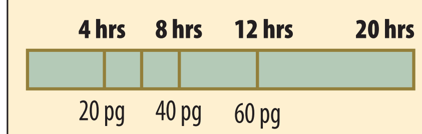
If it takes me four hours to read 20 pages, how many pages will I read in 12 hours?



Write in the number of pages read in 12 hours

Another question might be: if it takes me 4 hours to read 20 pages, how many pages will I read in 6 hours?

If it takes me four hours to read 20 pages, how many pages will I read in 6 hours?



Write in the number of pages read in 6 hours

Wits Maths Circle Problem set 16

There are two identical glasses. The first glass contains white grape juice, and the second glass contains the same amount of red grape juice.

A small amount of white grape juice is poured from the first glass into the second glass, making a mixture.

An identical amount of this mixture is then poured back from glass 2 into glass 1.

Which of the mixtures in the glasses is now more "contaminated" — the mixture in glass 1 or the mixture in glass 2?

Looking for patterns

Teaching through play is helpful

'Click click tap tap stamp click click clap clap stamp" This was how Professor Mike Askew from Wits University began the afternoon session for the group of 40 primary teachers.

He had neither welcomed the teachers back from lunch nor announced the topic he was planning to focus on that afternoon. Instead, he just continued clicking, tapping, clapping and stamping in the cycle referred to above and, as teachers started to turn and watch, he invited them to join in.

Nobody had been told how to join in, but we all did, and soon we were all in unison: "Click click tap tap stamp click click clap clap stamp"

When Mike asked how we knew what to do, teachers responded that they had recognised that Mike was repeating a sequence of five actions, and had imitated these actions with him.

Then he repeated his cycle of actions, this time with a count of the actions "1, 2, 3, 4, 5" accompanying each action. As he repeated this cycle, he continued the count, moving on to "6, 7, 8, 9, 10" and keeping track of the beat of his actions. Again, he invited teachers to join in and we all now counted our actions with him, getting up to about 25. Mike stopped us. "So what are you going to hear on the hundredth beat? A click, a tap or a stamp?" he asked.

To find an answer, some teachers replayed the actions in their heads; others replayed the actions physically quietly to themselves, whispering the count. Some started calling out: "It's a stamp. Stamp."

Mike waited to get more participants giving an answer. Most were now saying that the hundredth beat would be a stamp. "How do you know? I only saw you counting to about 20 or 22. How do you know what happens at the hundredth beat without actually getting there?"

Volunteering

A teacher volunteered this answer: "Every fifth beat is always on a stamp." Mike repeated this statement, laying careful emphasis on the words: "Every fifth beat."

"So you have used the structure of a pattern that repeated to make a prediction about what happens going forward?" A lot of teachers agreed. "So what would happen on the 120th beat?" Mike asked.

We could hear muttered exchanges involving "100" and "20" before teachers agreed that as 120 was also a multiple of 5; the 120th beat would also be associated with a stamp. "So what about the 124th beat then?" Here, many of us reverted to the actions, some of us beginning with a stamp for 120, and others starting with a click for 121.

Continuing from either of these points, we agreed that there would be a tap at the 124th beat. "How did you work that out?" Mike asked. One teacher said: "It's four beats on from 120" and another teacher followed up with, "It's four beats on from a multiple of five".

"So it is true for any multiple of 5, not just 120?" Mike ventured. "Yes, yes," many teachers agreed. The activity continued with groups of teachers making up their own click-tap-stamp patterns, sharing them with the group and making sure, firstly, that all could join in. Then they would ask what would happen in "far" positions of this sequence of repeated actions.

One group made a repeating pattern involving eight beats and teachers acknowledged that working out what would happen in far positions was harder because multiples of eight were harder to work out beyond 80 than multiples of five.

The activity is inclusive

All of us were able to join in, recognise and reproduce the action sequences that we heard. When it came to predicting what would happen in a distant position — on the 98th beat or the 145th beat, different teachers worked in different ways. Some recorded the pattern with abbreviations on paper like in the diagram on the previous page.

Others clicked, tapped and stamped while saying the numbers associated with these beats. Still others were seen just concentrating hard on an imagined replay of the actions and the multiple structure underlying it, before giving an answer.

Teachers, like children, came to see the multiple structure underlying repeating patterns in a range of different ways. By the end of the session, they were able to predict what would happen beyond their actions, but still at different rates and through different scaffolding processes.

Mike's skill lay in his ability to deal with this range, with a strong sense of where he was going, and with attention to supporting teachers to revoice their explanations in increasingly general ways.

Patterns

Pattern continuing and pattern finding activities can infuse a range of activities across the curriculum in primary mathematics and activities like the one that Mike introduced can be played by younger children in the early years.

Number patterns

Record the pattern by writing abbreviations on paper

C	C	T	T	S	5
C	C	T	T	S	10
C	C	T	T	S	15
					.
					.
					85
					90, and so on

Graphic: JOHN McCANN Source: PRIMARY MATHS AT WITS

Wits Maths Circle Problem set 17: Number sequence puzzles

1 What would happen in the 200th position in this action sequence?

CLAP CLAP STAMP JUMP JUMP CLICK
CLAP CLAP STAMP JUMP JUMP CLICK ... ?

2 What would the next two numbers in each sequence be?

A 1; 4; 7; 10; 13; 16; 19; ?; ?

B 1; 3; 6; 10; 15; 21; 28; 36; ?; ?

C 1; 1; 2; 3; 5; 8; 13; 21; ?; ?

D 2; 3; 3; 5; 10; 13; 39; 43; ?; ?

E 77; 49; 36; ?: ?

Graphic: JOHN McCANN

Teaching primary math

What is it that we want our learners to be able to do at the end of a lesson, or a term, or a year after our mathematics teaching?

We want learners to be able to answer the questions that we set them and to be able to answer the questions that appear in their exams. However, correctly answering a range of questions from different sources requires teachers to pay attention to more than simply guiding learners towards correct answers in classrooms.

Professor Mike Askew, an international expert in primary mathematics teaching and learning, at Wits School of Education, suggests three "actions of proficiency" that teachers should build into their mathematics lessons: fluency, reasoning and problem- solving. (See diagram The three actions on the right.)

Fluency

Fluency refers to recall of basic number bonds, core skills such as adding 10 or 100 to any number, and definitions such as "odd" and "even" numbers. In foundation phase, this includes helping children to practice and then memorise facts related to number bonds to 10 and later to 20.

Reasoning

Reasoning involves thinking about relationships between ideas as well as explaining this thinking. For example, inviting children to explain why adding two odd numbers will always give an even number and allowing children to use words and diagrams to show their thinking can provide opportunities for practising reasoning.

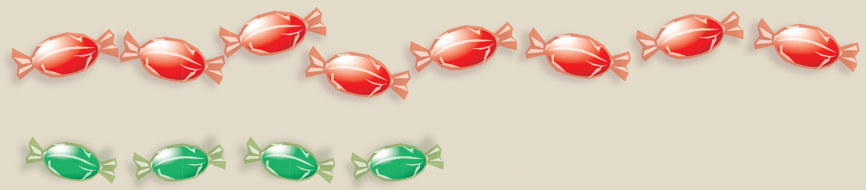
Children can also be encouraged to use known facts to work out new problems. For example, using the fact that 6 + 6 = 12 to work out the answer to 6 + 7, without having to add 6 objects to 7 objects, also supports the development of reasoning. As teachers, we might want to encourage a class to compare 6 + 6 with 6 + 7. We can question learners about what has changed and invite them to visualise and explain this change and its impact on the answer to the second problem.

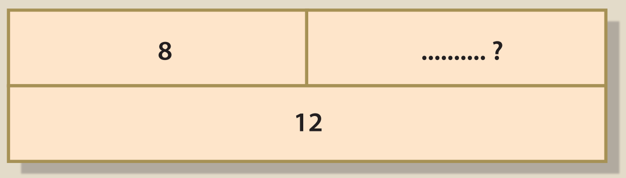
Demonstrating proficiency

Acting out a maths problem

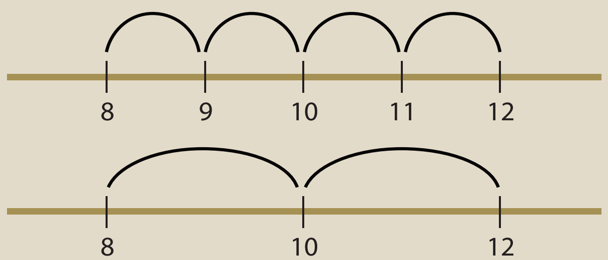
The actions are first represented visually and then move on to more concise or "compressed" representations.
Professor Mike Askew encourages teachers to help children to "act out" the problem — one child was Shalati and she picked up eight counters to represent her sweets (you can use real sweets). Another child had to give her more sweets (use a different colour) while the class kept track of how many sweets Shalati had in total.

Problem: Shalati has eight sweets. She is given more sweets and ends up with 12 sweets. How many sweets was she given?





What number is added to eight to equal twelve?



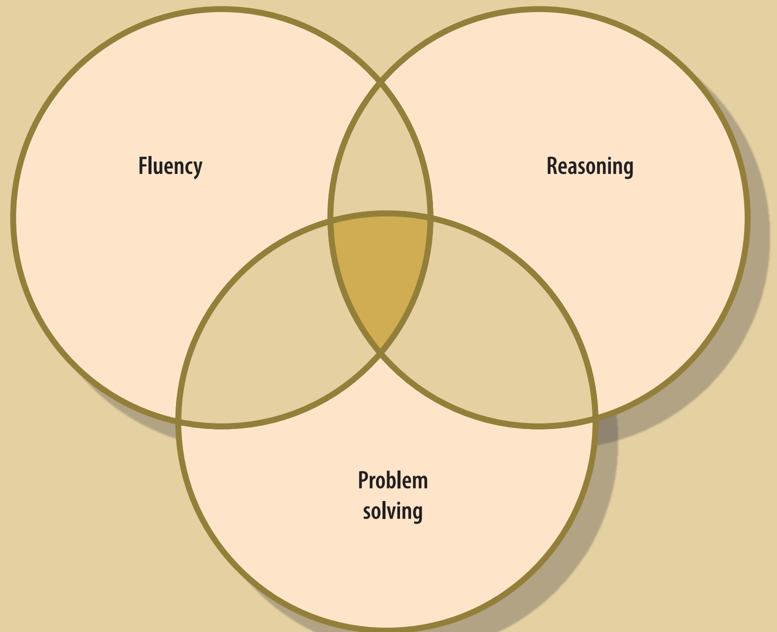
Fluency increases across these different diagrams and eventually sums like 8 + 4 can be answered by recall.

Graphic: JOHN McCANN Source: MIKE ASKEW

By reasoning their way to the answer, rather than reverting to counting, children can build a network of connected information about number and calculating. This network can then support the building of a bigger bank of recalled facts. This is vital for developing efficiency, because we do not want children to always go back to concrete counting to work out answers, especially when the number range they are dealing with starts growing bigger.

The three actions

of proficiency



Graphic: JOHN McCANN Source: MIKE ASKEW

Problem-solving

The final aspect that Askew mentions is problem-solving. Problem-solving involves being able to represent problems in words, through acting them out and in diagrams. It also involves choosing the correct operations to solve a problem and being able to communicate this process.

A demonstration

Askew demonstrates the opportunity for all three actions of proficiency through this problem: *Shalati has 8 sweets. She is given some more sweets and ends up with 12 sweets. How many sweets was she given?*

He encourages teachers to help children act out the problem. In other words, one child acts out being Shalati. She picks up 8 counters to represent her sweets. Another child acts out giving her some more sweets (in the form of counters in a different colour) and, meanwhile, the class is asked to keep track of how many sweets Shalati has in total. Representing these actions can begin with a diagram and then move on to increasingly compressed representations, as shown in the diagrams on the previous page. Fluency increases across these different diagrams and eventually sums like 8 + 4 can be answered by recall without the need for any diagram. However, fluency cannot develop without the support of the reasoning and problem-solving that are built into the actions and represented in the pictures above. Also, these representations provide children with resources that allow them to reason and solve problems as the number range gets higher and the problems become more complex.

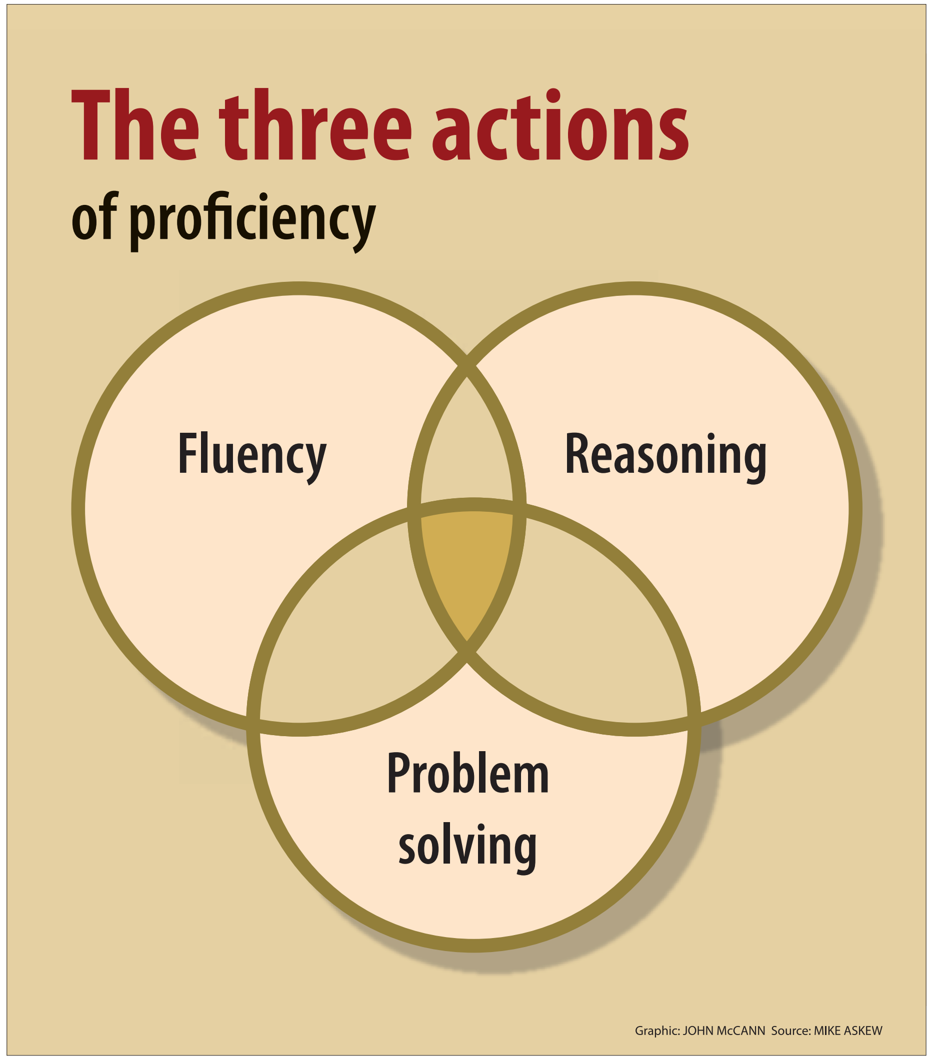
Reasoning and problem-solving

The communication skills that are built into reasoning and problemsolving allow teachers to hear children express their current thinking and therefore to make decisions about the kinds of representations they are ready to grasp. In this way, teachers can help children to progress towards more complex ideas. Research shows that children enjoy the challenges associated with fluency, reasoning and problem-solving. As mathematics teachers, we need to provide opportunities for children to engage in all three aspects if they are going to become mathematically proficient.

Problem solving and sense-making

We have previously looked at three "actions of proficiency that international primary mathematics expert Professor Mike Askew suggests teachers should build into their mathematics lessons: fluency, reasoning and problem solving.

In this article, we focus specifically on problem solving. Too often problem solving is just seen as another way of saying "word problems". And too often word problems come at the end of a chapter in the textbook. So if you've just done a chapter on division, you can be pretty sure that in each of the word problems you're going to have to use division. And our experience in classrooms shows us that children often simply identify the larger number and divide it by the smaller number, without any need to make sense of the story. When mixed word problems are given, one of the strategies we've seen used to help learners solve them is to look for key words — for example, if you see "more" then you need to add the numbers in the problem. Although tricks like this might help learners to answer some standard textbook problems, things go wrong quite quickly, as this example shows us: Lynn has seven more marbles than Nomsa. If Lynn has 15 marbles, how many marbles does Nomsa have? In cases like this, children who have been taught to identify key words, and use them to "cue" particular operations, get the wrong answer.



Graphic: JOHN McCANN Source: MIKE ASKEW

Sense-making in problem situations

Tricks based on this kind of "ignore the story" technique lead to a more broadly problematic discouragement of sense-making in problem situations. This is beautifully illustrated in research reported by Kurt Reusser where 97 grade one and two children were faced with the question: There are 26 sheep and 10 goats on a ship. How old is the captain? Reusser reported that more than three-quarters of the children gave the answer as 36. If the way we teach word problems and problem solving leaves learners believing that maths doesn't need to make sense, and that thinking about problems and how to tackle them is not worthwhile, then we are in trouble. So perhaps the most crucial idea is to ensure that problem solving is not taught using tricks, but linked strongly and clearly to sense- making. This can start right from the early years. Bringing the word problems from the end of the chapter to the beginning of the chapter can help to tap into learners' informal methods. Encourage learners to act out the problem. In our story with Lynn and Nomsa and the marbles, one child can be Lynn and one can be Nomsa. Both can have paper bags and teachers can help learners to read the question and add the information that is known to the paper bags that Lynn and Nomsa are holding. Lynn's bag would get the number 15 written on it, and the teacher can emphasise that Lynn has seven more than Nomsa. Leading questions and suggestions for deciding how to solve the problem might be: So does Nomsa have MORE or LESS marbles than Lynn?

Can we draw a picture showing what is in Lynn's bag?

Learners can be supported to make their own drawings and to try different approaches. As teachers, we should make sure they can explain their answers.

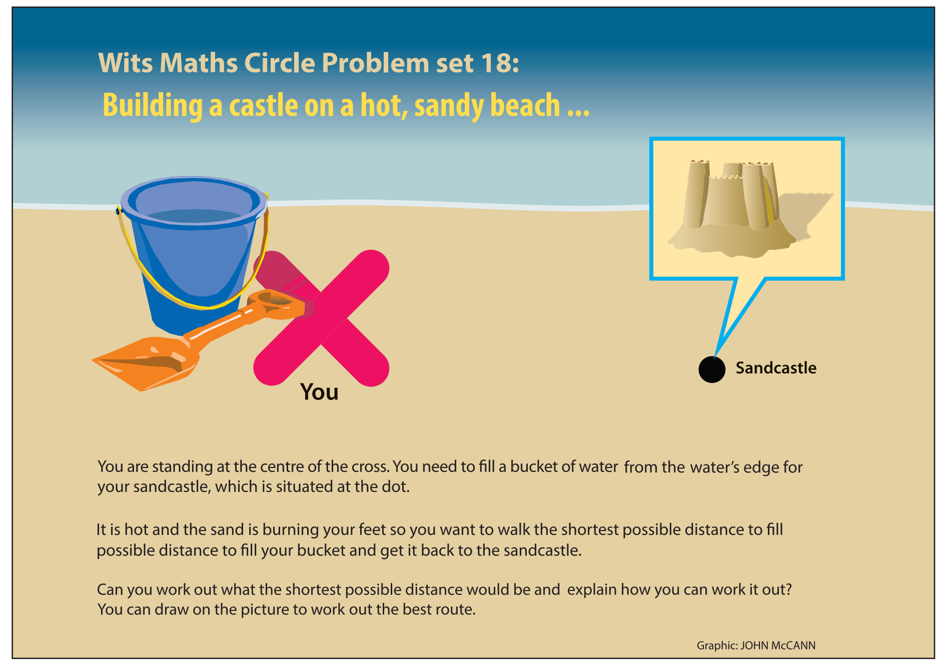
Giving experience of problems

And equally importantly we need to make sure that we give learners experience of problems that take effort and persistence to solve. So much of the mathematics we involve learners in has instant answers or at least an instant obvious route to the answer. If that is all we offer to learners they might never know that solving real problems in mathematics (as well as in life) requires one to wrestle for a sustained period: we go down one path only to find no solution and so have to try another way, look for patterns and possibilities, try different representations and work backwards as well as forwards. As with any mathematical skill, problem solving takes practice. And because genuine problem solving can be tough we need to create the kind of classroom environment that encourages children to take up the challenge and to persist despite the lack of instant gratification.

Problem solving

Among the ideas we've encountered in schools for creating a culture of mathematical problem solving are the following:

- Devoting one lesson a week to a problem solving session. Learners work in groups on substantive problems and write up their solutions to share with the rest of the class.
- Giving learners a problem of the week to work on at home. Interesting solutions are put up on a noticeboard for everyone to see. In some schools we know this has been encouraged as a family activity – with all members of the learner's family getting involved in trying to solve the problem.
- Having a box with a variety of problems written on cards. Learners are encouraged to take a card to work on if they finish their classwork or simply to take a card to work on at home if they would like to.
- One school we know has a regular maths problem solving picnic. Learners are divided into groups. Each group plans and brings picnic food and learners sit in groups around the school having their picnic and working on a series of mathematics problems.
- Asking learners to pose their own problems, perhaps for learners in the grade below. Here they have to think about choosing numbers, structuring and using words carefully, to provide clarity for younger learners.



Graphic: JOHN McCANN

Through games and fun

Learning Mathematics through games

Because children enjoy playing, some well-thought out games can be wonderful tools for making the maths class fun while engaging learners in mathematics.

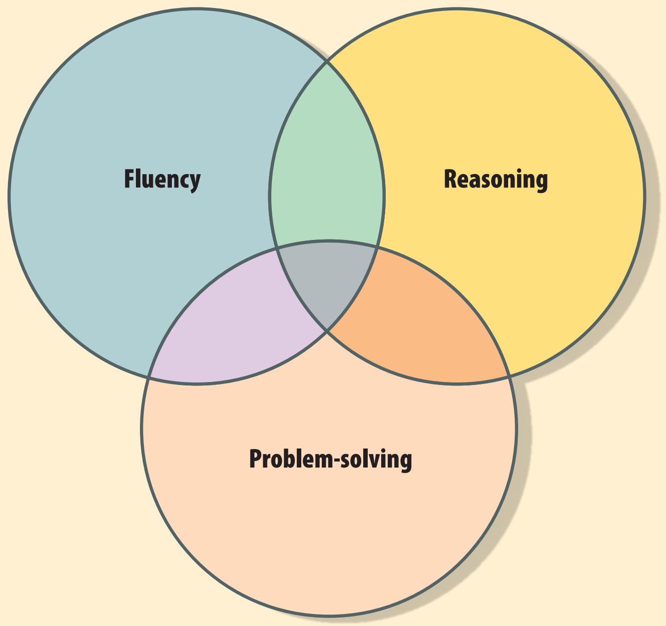
Games can be incorporated into the maths classroom either as a fun starter to a lesson to provoke the children's curiosity, or for the maths lesson scheduled during the last period of the day when just a little more is required to capture the learners' attention.

The games can also be very useful for afterschool maths clubs, fun maths evenings at school or for special maths days where teachers can promote maths and enthuse learners, teachers of other subjects, as well as parents, about maths and the enjoyment it offers.

We believe that in our maths classrooms we should be aiming to promote three key actions associated with mathematical proficiency: fluency, reasoning and problem-solving. Here we share some games you can use to do this.

Keys to being good at maths

The three most important abilities



Graphic: JOHN McCANN

Fluency

Maths 24: In this game, each learner receives a card with four numbers on it and is required to use all four numbers, once only, together with addition, subtraction, multiplication and division to make the number 24.

Create a pack of Maths 24 cards and learners can either work on them individually or in groups and can race against each other to see how many cards they can complete in a certain time.

(You can download a printable set of cards from Steve Sherman at *http://www.ru.ac.za/media/rhodesuniversity/content/sanc/documents/ Maths%2024%20-%20cards.pdf*)

Reasoning

Largest fraction: This game can be played with any number of players – although a group of about four players is probably ideal. Take a pack of cards. Remove the jokers and the face cards. Deal four cards to each player. Each player must choose two of their cards and make the largest possible proper fraction that they can by using one card as the numerator and the other as the denominator. The player with the largest fraction wins and earns a point.

Problem solving

Nim: The game Nim is an ancient game and variants of it have been played all over the world for centuries. We offer a simplified version of it for the classroom:

Make a pile of seven stones on the table. There are two players in this game. They must each take turns to remove either one or two stones from the pile. The player who removes the last stone is the loser.

Let the learners play the game a few times. Then make sure learners are starting to think about what they need to do in order to win the game. Does it matter who goes first? Does it matter how many stones you take first?

Let them hypothesise and then investigate their hypothesis. Once they have figured out the seven stone game, challenge them to investigate what would happen with a different number of stones.

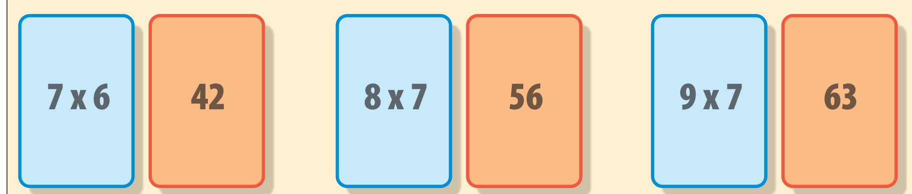
The successful use of any game in a classroom situation requires advance planning. It is essential to think about the practicalities such as the equipment needed, space required, noiselevel generated and so on.

It is equally important to have a clear mathematical objective when introducing a game. Is the purpose of the game to get learners to become increasingly fluent in doing basic calculations? Or is it to gain insight into the relative size of the numerator and denominator in determining the value of a fraction? Or is it to help them develop investigation and problem-solving skills? Carefully matching the game to your educational aims will help create an environment in which mathematics is learnt in a fun way.

Promoting maths fluency

A memory game

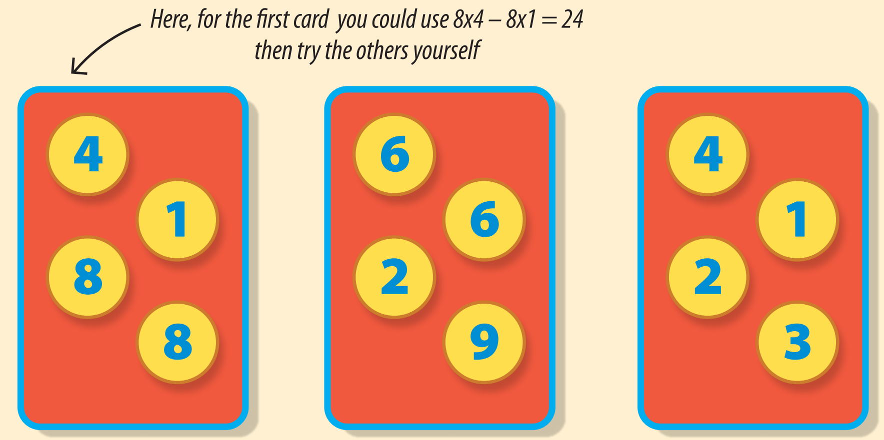
Make a set of matching pairs of cards where one of the pair contains a calculation and the other the solution. Here is an example:



The cards are then shuffled and laid upside down on a table. Learners take turns to select two cards to turn over. If they select a matching pair, they get to keep the cards. If not, they must turn the cards back over and the next learner takes a turn.

A speed game

In this game each learner receives a card with four numbers on it and is required to use all four numbers, once only, together with addition, subtraction, multiplication and division to make the number 24.



Create a pack of *Maths 24* cards and learners can either work on them individually or in groups. They can race against each other to see how many cards they can complete in a certain time.

You can download a printable set of cards from Steve Sherman at
www.ru.ac.za/media/rhodesuniversity/content/sanc/documents/Maths%2024%20-%20cards.pdf

Graphic: JOHN McCANN

Wits Maths Circle Problem set 19:

The following game, called Race to 20, is played with 2 people.

Person 1: Can choose to say 1, 2 or 3

Person 2: Chooses 1, 2 or 3 and adds it to person 1's number to get a new total.

Person 1: Chooses 1, 2 or 3 and adds it to person 2's total to get a new total.

Person 2: Chooses 1, 2 or 3 and adds it to person 1's total to get a new total.

Person 1: Chooses 1, 2 or 3 and adds it to person 2's total to get a new total. etc, etc

The first person who can make a total of 20 wins.

For example:

Person 1: 2
Person 2: 2 + 3 =5
Person 1: 5 + 1 = 6
Person 2: 6 + 3 = 9
Person 1: 9 + 1 = 10
Person 2: 10 + 2 = 12
Person 1: 12 + 2 = 14
Person 2: 14 + 1 = 15
Person 1: 15 + 1 = 16
Person 2: 16 + 3 = 19
Person 1: 19 + 1 = 20 and person 1 wins!

Question:

You are going to play this game. You can choose whether to be person 1 or person 2. Can you find a way to ensure that you will always win? Explain how.

Using home languages to support maths teaching

Teaching maths in the learners' home language can be used as a resource

In South Africa, language is often described as a "problem" or a barrier to successful learning. This view goes against international evidence that points to home language use at primary level supporting mathematics learning. In this article, we look at some ways in which language can be used as a resource in mathematics teaching in the early primary years.

In South Africa, national policy related to language encourages the use of home languages in the Foundation Phase. This links well to the evidence, that it is helpful to learn in the language that one speaks at home. Without this, children are often in a situation where they are faced with the difficulty of trying to learn mathematics in a medium they do not yet understand or possess any fluency in.

Learning the number words in the correct sequence is very important in early number learning. But teachers working with young children know that this is not easy!

There are lots of words to remember, and they have to come in a specific order. When counting out objects, children have to keep track of the objects they have counted, the objects left to count, and the words they are saying alongside this tagging of objects.

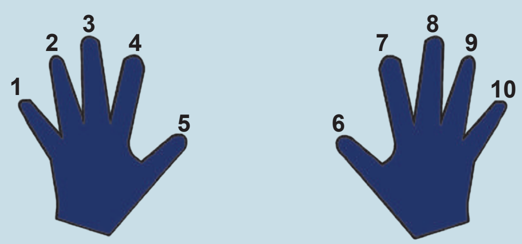
The number words in English are particularly difficult for a host of reasons. After learning the words from "one" to "ten", another set of words need to be learned to say the numbers from "eleven" to "twenty". After twenty, a more regular pattern sets in with: twenty-one, twenty-two, twenty-three, etc

Young children often get to "twenty-nine" but then say 'twenty-ten' because they know that ten follows nine. And similar problems come in after 'thirty-nine' and 'forty-nine' and so on. A new set of special words have to be learned for all the multiples of ten: twenty, thirty, forty, and so on.

In contrast to English, many of the South African languages have a much simpler and more regular structure. Look at the following table of numbers showing how to say the numbers from one to thirty in sePedi:

1	2	3	4	5
tee	pedi	tharo	nne	hlano
11	12	13	14	15
lesome tee	lesome pedi	lesome tharo	lesome nne	lesome hlano
21	22	23	24	25
masome pedi tee	masome pedi pedi	masome pedi tharo	masome pedi nne	masome pedi hlano

6	7	8	9	10
tshela	šupa	seswai	senyane	lesome
16	17	18	19	20
lesome tshela	lesome šupa	lesome seswai	lesome senyane	masome pedi
26	27	28	29	30
masome pedi tshela	masome pedi šupa	masome pedi seswai	masome pedi senyane	masome tharo



Neither of us are sePedi speakers, but we both began by learning the number names for one to ten. Beyond this, paying some attention to the patterns and structure that we can see emerging in this table allows us to 'construct' the number names for the next line. We can see the following patterns:

11 is "lesome tee" or, literally, "ten, one" (10 and 1, if we write this using the number symbols)
12 is "lesome pedi" which is "ten, two" (10 and 2)
13 is "lesome tharo", "ten, three" (10 and 3)

We can follow this pattern through to 19, and we then see that 20 is "masome pedi". Here, the prefix for some has changed from le- to ma-. The "ma" tells us that we have a plural number of tens, and the "pedi" that follows "masome" tells us that we now have two 10s, not one 10 as before. After 20, the same pattern seen before repeats itself:

21 is "masome pedi tee" – "twenty, one" (20 and 1), and so on.

Unlike in English, where a new word needs to be learned for 30, the pattern seen above continues in sePedi:

30 is "masome tharo" – "tens – three"

What is important is that we are paying attention to the patterns and structure of how numbers get built up in sePedi rather than trying to remember a long sequence of numbers.

This is important for teaching. We think that paying attention to the structure of number names in sePedi is the more important aspect to focus on after learning the number names for 1-20, rather than trying to teach children to learn long lists of number names. Learning how numbers are constructed allows children to create larger number names for themselves using the rules of the pattern. Tasks for focusing on these rules include questions like these:

a) 34 is masome ______ nne
b) Jabu says 46 is masome nne tshela
Thuli says 46 is lesome nne tshela
Who is correct — Jabu or Thuli?

We would use 10 sticks and unit squares to help children to see what these numbers look like as quantities — for example, 34 is shown below:

We enjoyed extending our learning of number names to an unfamiliar language. Our thanks to Manono Mdluli, a colleague and lecturer in the Wits School of Education for teaching us how to construct the numbers up to 99 (so far) in sePedi.

Wits Maths Circle Problem set 20

There are six numbers written in five different scripts

Can you sort out which is which?

Write 51 in each script

(source: nrich)

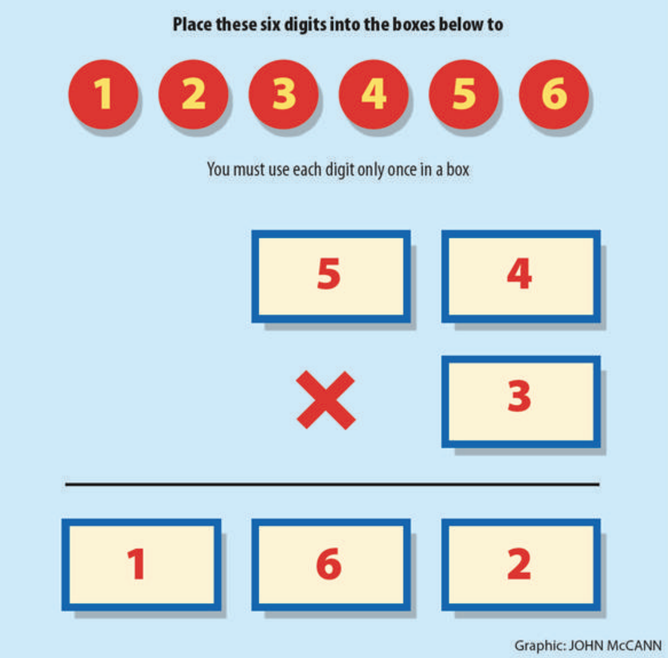
Solutions

Problem Set 1 Answers

1. 27 + 46 = 28 + 45 = 29 + 44 = 30 + 43 = 23 + 50
   We can see, for example, that because we add 1 to 27 we must take 1 away from 46 to keep the answer the same. The 'sum' or 'total' in an addition sentence stays the same if the increase in one quantity is balanced out by the decrease in the other quantity.
2. 642 + 531 or 641 + 532 or 632 + 541 or 631 + 542
   You need the two largest digits (6 and 5) in the hundreds position, the next two largest digits (3 and 4) in the tens position and then 1 and 2 in the units position to make the largest sum.

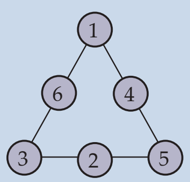
Problem Set 2 Answers

Multiplication



Problem Set 3 Answers

1.



2.

4	3	8
9	5	1
2	7	6

Problem Set 4 Answers:

1. 654 – 123 will give the greatest difference. We choose to put the largest digit and smallest digit in the hundreds place value to give the greatest difference in the hundreds position. The second largest digit and second smallest digit in the tens place value to give the next greatest difference in the tens position. And of the two digits left (4 and 3), we put the largest one in the starting number and the smaller one in the other number to give the greatest difference again.

2. This will take 12 trips. Let's talk about them going from shore A to shore B.
   Trip 1: Both children go from shore A to shore B.
   Trip 2: One child goes back to shore A.
   Trip 3: One adult goes from shore A to shore B.
   Trip 4: The child goes from shore B back to shore A.
   Trip 5: Both children go from shore A to shore B.
   Trip 6: One child goes back to shore A.
   Trip 7: The second adult goes from shore A to shore B.
   Trip 8: The child goes from shore B back to shore A.
   Trip 9: Both children go from shore A to shore B.
   Trip 10: One child goes back to shore A.
   Trip 11: The third adult goes from shore A to shore B.
   Trip 12: The child goes back to join the other child on shore A.

Problem Set 5 Answers

1a). Work backwards:

6 bananas left were the 2 remaining piles so each pile was 3 bananas. So when Moses divided the bananas into 3 piles there were 9 bananas. He had eaten one so there were 10 bananas when he got there.

10 bananas were the 2 remaining piles so each pile was 5 bananas. So when Milly divided the bananas into 3 piles there were 15 bananas. She had eaten one so there were 16 bananas when she got there.

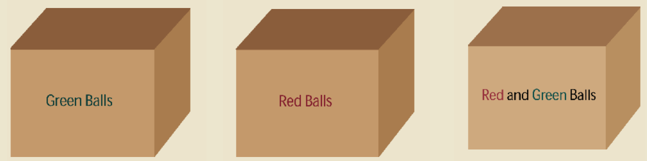
16 bananas were the 2 remaining piles so each pile was 8 bananas. So when Mandi divided the bananas into 3 piles there were 24 bananas. She had eaten one so there were 25 bananas when she got there.

b) Mandi got 2 bananas, she ate 1 and she hid 8 so she got 11 bananas.
Milly got 2 bananas she ate 1 and she hid 5 so she got 8 bananas.
Moses got 2 bananas, he ate 1 and he hid 3 so he got 6 bananas.

2. We know 326 × 17 = 5542. So 326 × 18 means we have one more group of 326 i.e. the answer will be 5542 + 326 = 5868.

Problem Set 6 Answers

1.



Because we know the boxes are incorrectly labeled we know the following:
Box 1 must contain red balls only or red and green balls
Box 2 must contain green balls only or red and green balls
Box 3 must contain red balls only or green balls only

So if I reach in to box 3 I will pull out either a red ball or a green ball.

If I pull out a red ball then I know
- Box 3 contains red balls only
- So then box 1 must contain red and green balls
- And box 2 must contain green balls only

If I pull out a green ball then I know
- Box 3 contains green balls only
- So then box 2 must contain red and green balls
- And box 1 must contain red balls only.

I have to choose box 3 to reach into, as this is the only box where pulling one ball out tells me for sure what is actually in the box. This does not happen if we reach in to either of the other two boxes.

2.

	In the 8 liter jug	In the 5 liter jug	In the 3 liter jug
Start	8	0	0
Pour from 8-jug into 5-jug	3	5	0
Pour from 5-jug into 3-jug	3	2	3
Pour from 3-jug into 8-jug	6	2	0
Pour from 5-jug into 3-jug	6	0	2
Pour from 8-jug into 5-jug	1	5	2
Pour from 5-jug into 3-jug: 4 left in 5-jug	1	4	3

Problem set 7 answers:

1) a. Since b and c are both less than 1 when we multiply them together the answer will be smaller than c and smaller than b.

2a) *3,1 × 6,9 will be close to 3 × 7 = 21 so it will be closest to 20.*
b) 0,31 × 6,9 will be close to 0,3 × 7 so it will be closest to 2.

Problem Set 8 Answers

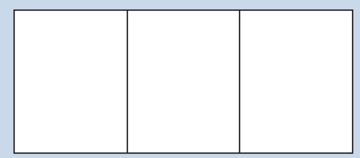
Looking at the first line of stars we could see that C × "ABC" must be a four digit number, so C could not be 1. We then needed to check whether C was 2, 3, 4, 5, 6, 7, 8 or 9. As this left us with a lot of options to consider for C, we moved on to look at what we could deduce from the second line of stars.

Looking at the second line of stars we could see that "ABC" x A must be a 3-digit number. This only happens if A is 1, 2 or 3. We also know the product, A × C, must end with the digit A. So we worked through whether this could happen for A = 1, 2 or 3 (keeping in mind that we know C can't be 1).
- We can't have A =1, since 1 × C = C and A and C can't be the same number.
- It is possible to have A = 2 as long as C = 6, since 2 × 6 = 12
- We can't have A = 3 because if we look at the first 9 multiples of 3 (3; 6; 9; 12; 15; 18; 21; 24; 27) none of them end in 3 except the first one – and we know C can't be 1.

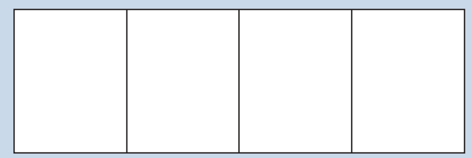
We then had A = 2 and C = 6 and we just needed to figure out the value of B.
Moving to the last line of stars, we deduced that for "ABC" x B to be a 4-digit number, B had to be 4 or bigger since "ABC" was a 'two hundred and something' number. We also needed the B x C i.e. B × 6 product to end with the B digit. There are only two options for this: B=4 or B=8. But ABC = 246 would not give a 4-digit number in the last line of stars, so B must =8. So the problem must be 286 x 826.

Problem Set 9 Answers

1. The rectangle shown is 3 equal-sized pieces taken from the original sheet of paper which was made up of 4 of those pieces. So we can divide the rectangle into 3 pieces:

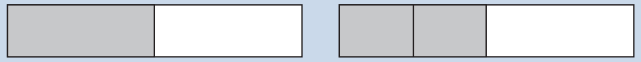


And then add an extra one of those pieces on to get the original sheet of paper.

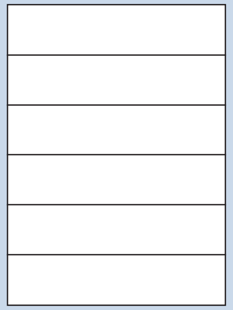


2. You cannot compare fractions if the "wholes" you are comparing are different. Clearly ½ of a huge bar of chocolate will be more than ½ of a small bar of chocolate. You need to compare fractions of the same thing. So to compare $^1/_2$ and $^2/_4$ you need to take something and divide it into 2 pieces and

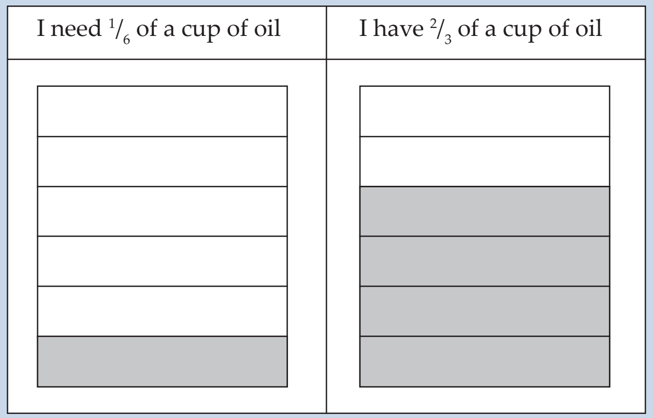
shade one and then take the same thing and divide it into 4 pieces and shade 2.



3. I will use a shape like this to represent a cup measure



Then I can draw the amount of oil I need and the amount of oil I have as follows:



And so I can see that the amount I need is 1 part out of the 4 parts I have i.e. the amount I need for the recipe is $\frac{1}{4}$ of the oil I have in the bottle.

4. There are of course many story sums that will work here. One could be: I have $\frac{2}{3}$ litres of milk in my fridge. I use $\frac{1}{2}$ litre to make pancakes. How much milk (in litres) do I have left?
A common error students make is coming up with a story that actually leads to the calculation $\frac{2}{3} - \frac{1}{2}$ of $\frac{2}{3}$ e.g. I have $\frac{2}{3}$ of a packet of smarties and I eat $\frac{1}{2}$ of them. How many do I have left?

5a. This does lead to $\frac{2}{3} \times \frac{1}{4}$ because if you want to make $\frac{2}{3}$ of the recipe you'll need $\frac{2}{3}$ of all the ingredients so you'll need $\frac{2}{3}$ of $\frac{1}{4}$ of a cup of water.

b. This is not a story sum for $\frac{2}{3} \times \frac{1}{4}$. The $\frac{1}{4}$ tells us the proportion of cake we have.
The $\frac{2}{3}$ tells us what proportion of Mrs Watson's class want cake.
We can't combine the $\frac{2}{3}$ and the $\frac{1}{4}$ because they are fractions of two totally different things (the class and the cake). To find out what fraction of the cake each student would get we would need to know how many children are in Mrs Watson's class so we can find out how many children want cake (by calculating $\frac{2}{3}$ of the total number of children). Then we can divide the $\frac{1}{4}$ of the cake by that number of children to work out what fraction of cake they would each get. In this case, we cannot actually answer the question because we do not have all the information we need as they don't tell us how many children are in Mrs Watson's class.

Problem set 10 answers

1a) $\frac{16}{32} = \frac{1}{2}$. So $\frac{16}{31}$ will be bigger than $\frac{1}{2}$ since the whole is cut into only 31 pieces (so these will be bigger than if the whole is cut into 32 pieces) and you take 16 of the pieces. Or you could think of it as $(15\frac{1}{2}) / 31 = \frac{1}{2}$ so $\frac{16}{31}$ will be bigger than ½

b) ¾ × 4 = 3. It is useful to think of ¾ as the outcome of dividing 3 by 4, i.e. the same as 3 ÷ 4. Reversing this division process means that 3 must be four times bigger than ¾.

c) $\frac{8}{9}$ is close to 1 and $\frac{10}{11}$ is close to 1 so $\frac{8}{9} + \frac{10}{11}$ will be close to 2.

2. The first guess we could make is 2.498. That uses an 8 and is pretty close to 2.5. However I can immediately think of a better answer 2.4998 that gets that little bit closer. And then 2.49998 is an improvement on that. And 2.499998 is an improvement on that and 2.4999998 is an improvement on that and so on and so on. This tells us there is no decimal number not using the digit 5 and using the digit 8 with a finite number of decimal places that is as close to 2½ as possible.

In a similar way you can see there is no decimal number that isn't 1 that is as a close to 1 as possible. 0.9999999 is very close. But 0.99999999 is even closer!

Problem set 11 answers

Adding fractions

The answer to this activity is less important than the exploration and thinking that goes into playing with the possibilities. So, for example, I know I can't make the numerator bigger than the denominator in either fraction or else it'll already be bigger than 1. I can figure out that if I choose ½ as the first fraction, I know I need to make the second fraction just a little less than ½ so $\frac{3}{7}$ may be a good choice. Thinking through similar ideas about fractions allows me to explore the possibilities that get me close to 1.

The nearest I could get to 1 was by starting with $\frac{5}{6}$. This is just $\frac{1}{6}$ short of 1 so I knew if I added $\frac{1}{7}$ (which is smaller than $\frac{1}{6}$) I would still have less than 1.

$$\frac{5}{6} + \frac{1}{7} = \frac{6 + 35}{42} = \frac{41}{42}$$

Graphic: JOHN McCANN

Problem set 12 answers

The strip below represents all the senior learners and the shaded bit represents those who have partners.

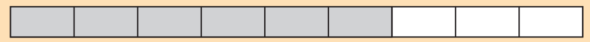
Now since $\frac{2}{3}$ of the senior learners $= \frac{3}{5}$ of the junior learners that shaded bit also represents $\frac{3}{5}$ of the junior learners.
If I take the strip representing the total numbers of senior learners and divide each of its blocks into 3 pieces it looks like this:

Now as the pieces below represents $\frac{3}{5}$ of the junior learners.

I can see that 2 little blocks = $\frac{1}{5}$ of the junior learners so to get something that represents all the junior learners I need 5/5 i.e. I need 10 little blocks.

So now I can see that I can represent all the learners at the schools as follows:

Senior learners:



Junior learners:

All the little blocks are the same size and shaded blocks represent those who have partners . I count 19 blocks in total of which 12 are shaded so the fraction of learners who have partners is 12/19.

Problem set 13 answers

Working out the remainder, an amount left over after dividing numbers

Question one

Yes, Lerato and Bongani could be thinking of the same number
One is a possible answer. The other possible answers are one more than a common multiple of four and seven, for example, 28 + 1 = 29 or 56 + 1 = 57.

Question two

Let the number be N ...
if it leaves a remainder of four when divided by five, then if we add one to it, it will be divisible by five.
if it leaves a remainder of three when divided by four, then if we add one to it, it will be divisible by four.
if it leaves a remainder of two when divided by three, then if we add one to it, it will be divisible by three.
if it leaves a remainder of one when divided by two, then if we add one to it, it will be divisible by two.

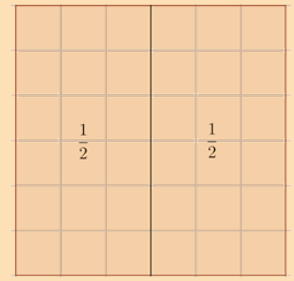
If we add one more to the number, then it will be divisible by five, four, three and two.
N + 1 is divisible by five, four, three and two.
Therefore one possibility for N + 1 is 5 x 4 x 3 x 2 = 120, which means that N would be 119.

Graphic: JOHN McCANN Quiz source: NRICH

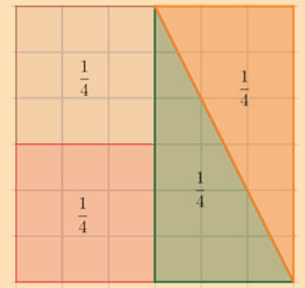
Problem set 14 answers

If we look at how the picture was made up we see the following:

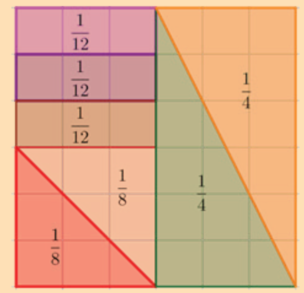
First the square was cut into half so we get



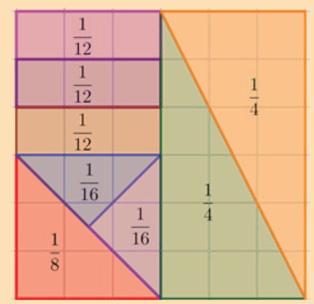
Then each of those halves was cut in half and $\frac{1}{2}$ of $\frac{1}{2} = \frac{1}{4}$



We cut the top left $\frac{1}{4}$ into 3 pieces and $\frac{1}{3}$ of $\frac{1}{4} = \frac{1}{12}$
And we cut the bottom left $\frac{1}{4}$ into 2 pieces and $\frac{1}{2}$ of $\frac{1}{4} = \frac{1}{8}$
So we get



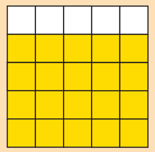
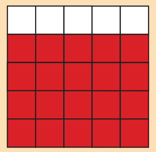
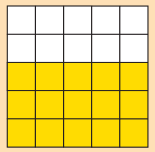
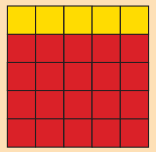
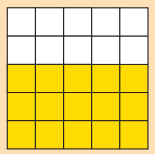
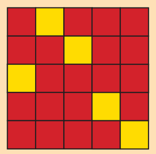
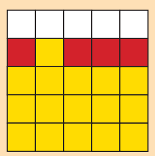
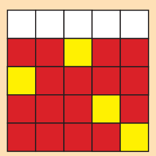
And finally we cut the $\frac{1}{8}$ into 2 pieces and $\frac{1}{2}$ of $\frac{1}{8} = \frac{1}{16}$ so we have:



Problem set 15 answers

If 40 oranges are worth 60 apples, then 100 oranges are worth 150 apples.
If 75 apples are worth 84 peaches, then 150 apples are worth 168 peaches
If 100 peaches are worth 1 box of grapes, then 168 peaches are worth 1,68 boxes of grapes.
And if 3 boxes of grapes are worth 40 pounds of pecans then 1,68 boxes of grapes are worth 40 ÷ 3 × 1,68 = 22,4 pounds of pecans.

Problem set 16 answers

Glass 1	Glass 2	
		Picture each glass of juice divided up into 20 units.
		Now imagine we pour 5 units of the white juice into the red juice. We now have 25 units in glass 2, 5 of these are white juice and 20 units are red.
		Mix well. In glass 2: we know that in the juice mixture there are 5 units of white and 20 units of red. This means that for every 1 unit of white there are 4 units of red. So in any 5 units of mixture there is 1 unit of white and 4 units of red.
		Pour 5 units of juice from glass 2 back into glass 1. See that we have 16 units white and 4 units red in glass 1. And 16 units red and 4 white in glass 2.
So each glass is equally contaminated		

Problem set 19 answers

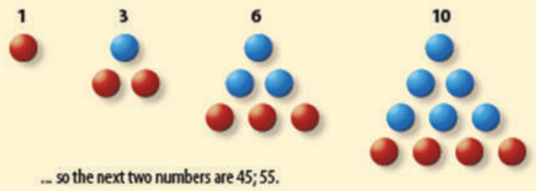
To win, I must say 20.
In order for me to say 20, my partner must say 17, 18 or 19.
If I say 16 then I force my partner to say 17, 18 or 19.
In order for me to say 16, my partner must say 13, 14 or 15.
If I say 12 then I force my partner to say 13, 14 or 15.
In order for me to say 12, my partner must say 9, 10 or 11
If I say 8 then I force my partner to say 9, 10 or 11
In order for me to say 8, my partner must say 5, 6 or 7
If I say 4 then I force my partner to say 5, 6 or 7
In order for me to say 4, my partner must say 1, 2 or 3
So if I get my partner to start and then I say 4, 8, 12, 16, 20, I will win.

Problem set 17 answers

1. At every multiple of 6 (6, 12, 18, and so on) you will be at the end of a sequence and the clicking. At 198, which is a multiple of 6, you will be clicking. Then 199 will be a clap and so will 200.

2.
A. In this sequence, you add 3 to get the next term. So the next two numbers are 22; 25.

B. This sequence gives what are called the triangular numbers. They can be pictured like this ...



... so the next two numbers are 45; 55.

C. This is the Fibonacci sequence of numbers. Here each term is the sum of the previous two terms, 1+1 = 2; 1+2 = 3; 2+3 = 5; 3+5 = 8, and so on. So the next two numbers are 13 + 21 = 34 and 21+34 = 55.

D.

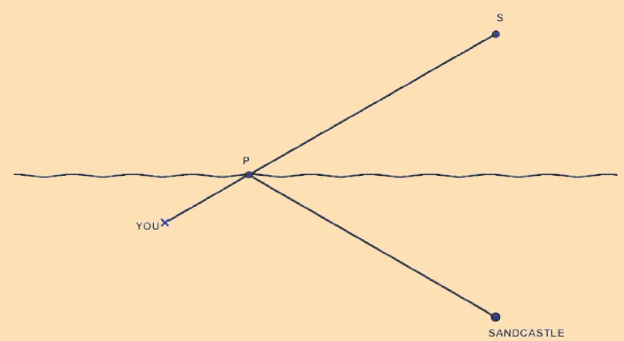
+1	×1	+2	×2	+3	×3	+4	×4	+5	
2	3	3	5	10	13	39	43	172	177

E. 7×7 = 49; 4×9 = 36; 3×6 = 18; 1×8 = 8

Graphic: JOHN McCANN

Problem set 18 answers

Draw a point S at the same place behind the water's edge as your sandcastle is in front of the water's edge. A straight line gives you the shortest distance between two points so draw a straight line from you to point S. You must aim for the point P where this line crosses the water's edge. The path from you to the sandcastle will be the same length as the path from you to S, which we know is the shortest.







Problem set 20 answers

Assuming that each script used some form of place value system
૨ २ 二 ٢ all represent 2.

Using this we can draw up a table that contains the numbers in the script we recognise in column A, and the different versions of two alongside:

A	B	C	D	E
2	٢	૨	२	二
13				
25				
58				
83				
100				

Knowing the 2s, we know these icons should appear in the tens position of each script's version of 25. For example, if we start with the script where ٢ is 2, we can see from the list in column A that we should have a 25 that starts with ٢. Looking through the grid we find ٢٥. So the "upside down heart" is 5.

Knowing the 5 we can look next for a 58 containing the 'upside down heart' in the tens position. We find ٥٨. So the "upside down V" is 8 and then ٨٣ is 83. As we know what 3 looks like, we can see that ١٣ is 13. And finally knowing what 1 looked like, we know that ١٠٠ must be 100, completing column B.

Using the same process, the other columns can be filled in.

The script that is slightly different is the one that starts with 二. Here 三 is '3'. Looking for 13, which contains three units (and therefore should contain 三) and '25' (which should contain 二 in the tens position) we can find 十三 and 二十五. Both of these icons contain the 十. So by looking at the structure of these two numbers we see:
13 is made up of 1 ten and 3
25 is made up of 2 tens and 5
which points to 十 standing for '10'.
These are the Chinese numerals.





Lynn Bowie loves mathematics and teaching and has taught mathematics at all levels – from primary through to university. She has a PhD From Wits University, where she used to coordinate the programme for prospective primary maths teachers – and where her collaboration with Hamsa Venkat began. She now works for an NGO, OLICO Youth, (www.olico.org) in Diepsloot where she has been developing a free online practice tool for grade 7 – 9 Mathematics (learn.olico.org) and supporting Mathematics Clubs for primary school learners. She is a Visiting Associate at Wits University, where she continues her collaboration with Hamsa Venkat, with a particular focus on stimulating a love of mathematics and enjoyment of problem-solving in primary school teachers.

Hamsa Venkat loves mathematics, teaching mathematics and teaching mathematics teachers. She holds a Professorship and the SA Numeracy Chair at Wits, and leads a research and development project that aims to support the development of mathematics teaching and learning in ten government primary schools. She is also involved in work on Foundation Phase assessment and primary mathematics teacher education in the national terrain. She works with a team of postgraduate students and staff at Wits University on a range of interventions that are being developed, trialed and studied to assess their usefulness and practicality for broader implementation.

Hamsa taught mathematics in high schools in London prior to moving to South Africa, and has worked in mathematics teacher education and research in England and South Africa.