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TEACHING & LEARNING DEVELOPMENT CAPACITY IMPROVEMENT PROGRAMME

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PRIMARY TEACHER EDUCATION (PrimTEd) PROJECT

GEOMETRY AND MEASUREMENT WORKING GROUP

DRAFT FRAMEWORK FOR A TEACHING UNIT

Preamble

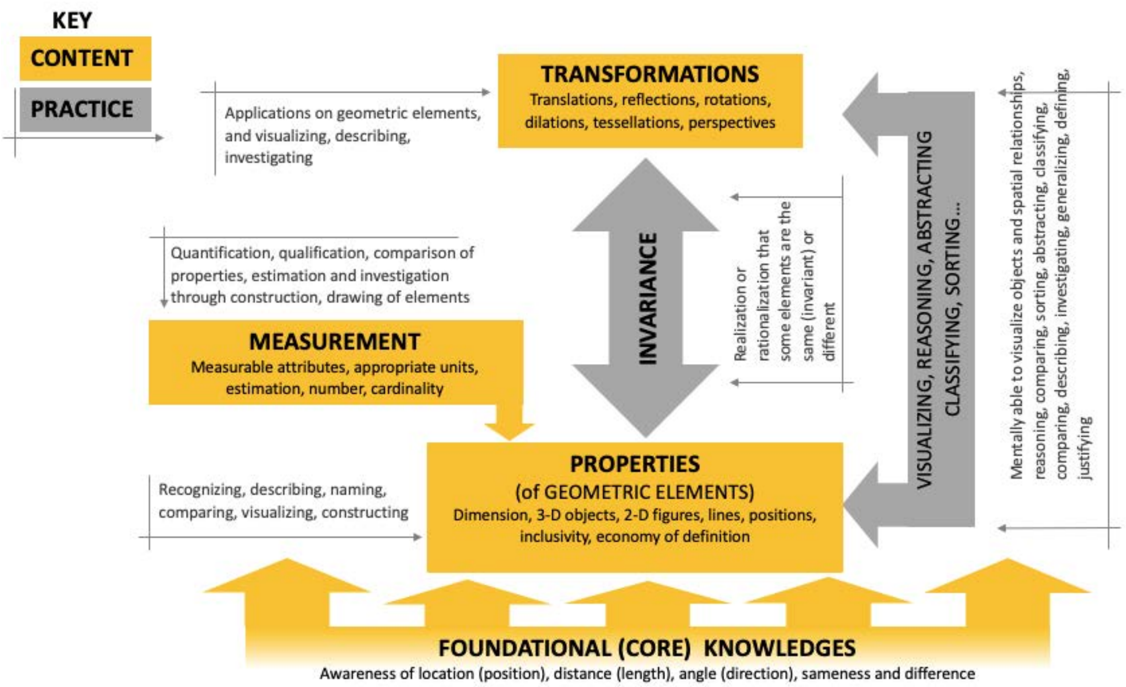
The general aim of this teaching unit is to empower pre-service students by exposing them to geometry and measurement, and the relevant pedagogical content that would allow them to become skilful and competent mathematics teachers. The depth and scope of the content often go beyond what is required by prescribed school curricula for the Intermediate Phase learners, but should allow pre-service teachers to be well equipped, and approach the teaching of Geometry and Measurement with confidence. Pre-service teachers should essentially be prepared for Intermediate Phase teaching according to the requirements set out in MRTEQ (Minimum Requirements for Teacher Education Qualifications, 2019). “MRTEQ provides a basis for the construction of core curricula Initial Teacher Education (ITE) as well as for Continuing Professional Development (CPD) Programmes that accredited institutions must use in order to develop programmes leading to teacher education qualifications.” [p6].

Target Audience

For utilisation by teacher-educators for the education of Intermediate Phase mathematics pre-service teachers.

“Big Ideas” in Geometry and Measurement

<u>What is a Big Idea?</u>



'Using big ideas as a focus helps teachers see that the concepts represented in the curriculum expectations should not be taught as isolated bits of information but rather as a network of interrelated concepts.'

Guide to Effective Instruction in Mathematics Kindergarten to Grade 3 – Measurement, 2007

Big ideas are fundamental mathematical concepts/constructs that permeate mathematics and cut across content areas creating a coherent perspective of the discipline. These big ideas also create a particular perspective of mathematics that can be utilized to solve problems by promoting unique strategies based on the big idea/s, strategies that would otherwise have been inaccessible.

Describe what a Big Idea is. What it is all about?

"'Big ideas and understandings' could be taken as a foundation for a primary and lower secondary mathematics curriculum. A 'big idea' is defined as a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understanding into a coherent whole."

Randall Charles

These 'big ideas' allow for a focus on particular properties and create unique structures according to which certain mathematical ideas or conceptions are constituted. This affords those learning mathematics to develop a particular 'gaze' which allows them to perceive mathematics as a coherent whole rather than a collection of disjointed principles or conceptions.

This section will address properties, measurement and transformations as constituents of the concept of geometry and measurement big ideas

Rationale

Why is this Big Idea so important?

As Measurement constitutes an aspect of the "big idea" as relates to the learning of geometry and measurement, it becomes pertinent to locate this aspect of the content within the context of this body of knowledge.

The intention is to produce materials that exhibit the interrelated nature of the concept measurement. The aim is to develop a coherent understanding of measurement that would allow for consistent implementation across the various measurement contexts.

How is it defined?

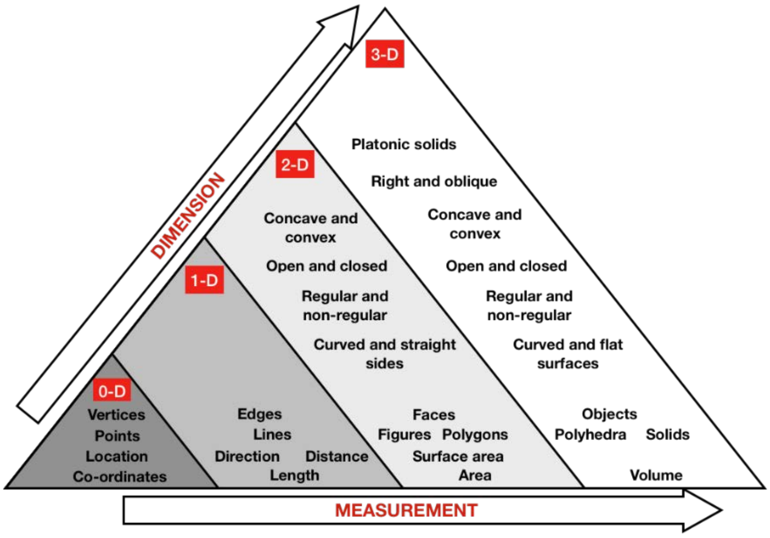
'A measurement is a ***number*** *that indicates a comparison between the* ***attribute*** *of an object being measured and the same attribute of a given* ***unit*** *of measure.'*

Van de Walle, 2015

This is the preferred definition that guides the development of the materials found within this resource. It is accepted that this is not the only definition for this 'big idea' or content area but common to most definitions of measurement are the following three aspects: attribute, unit and number. The interplay between these aspects allows for the development of a conception of what measurement is. This definition also allows for the transferability of the concept of measurement across applications of the topic, i.e. across 1-, 2-, and 3-dimension.

How will it help develop a deeper understanding of the many concepts dealt with in geometry?

Just as Properties reference learners being able to 'qualify' what geometric objects they encounter; Measurement allows learners to 'quantify' these geometric objects. Having a sound understanding of how these aspects of mathematics articulate to produce a realization of what geometry is.



How is it translated into the school curriculum?

'In building a program, teachers need a sound understanding of the key mathematical concepts for their students' grade level and a grasp of how those concepts connect with students' prior and future learning. They need to understand the "conceptual structure and basic attitudes of mathematics inherent in the elementary curriculum" (p. xxiv) and to know how best to teach the concepts to students. Concentrating on developing this knowledge and understanding will enhance effective teaching.'

Ma, 1999

Content Standards for Measurement

1. Recognizing the **attribute** being measured
   The ability to recognize and isolate the (measurable) attribute of the object being measured. The ability to recognize the extent of this attribute as the extent of the measurement.

2. Identifying a **unit**
   The ability to select a unit that correlates (dimensionally) with the attribute being measured.

3. **Cardinality** of the units employed
   Realizing that the total number of units constituting the extent being measured establishes the final measure.

4. **Iterating** units
   Realizing that a measure is constituted through iterating the selected unit across the extent of the attribute of the object being measured.

5. **Estimation**
   The ability to employ estimation as a means to demonstrate an understanding of units and the measurement process.

6. The relation between **Number** and **Measurement**
   Understanding that measurement mediates the relation between real-world contexts and number knowledge when attempting to quantify objects encountered in the environment.

Theories, Teaching Approaches and Methodology

Many children use measurement instruments or count units in a rote fashion and apply formulas to attain answers without meaning. For some attributes, children have difficulty establishing a proper unit for measurement, such as area (some erroneously count lengths) and volume (some count faces) (Clements & Battista, 1992).

Clements and Sarama suggest that measurement is the best way for young kids to learn about math. They go so far as to say it's better than counting. "We use length consistently in our everyday lives," they write. "[Measurement] can help develop other areas… including reasoning and logic. Also, by its very nature, [measuring] connects two critical domains of early mathematics: geometry and number." Non-numerical ordinal judgments may develop before the capacity for numerical ordinal judgments. (Douglas H. Clements and Julie Sarama, 2009)

Measure mediates between simple computational structures for non-numeric comparisons of aggregates on the one hand (core domain), and simple computational structures for computations with numbers, on the other hand (noncore domain). Measure is structure-preserving in such computational contexts (Ovendale, Brookes, Colletta & Davis; 2018).

Young children can be guided to have appropriate measurement experiences. They naturally encounter and discuss quantities (Seo & Ginsburg, 2004). They initially learn to use words that represent quantity or magnitude of a certain attribute. Then they compare two objects directly and recognize equality or inequality (Boulton-Lewis, Wilss, & Mutch, 1996).

Both inductive and deductive approaches should be modelled with this teaching unit. Teaching methodology should include investigation, and discussion, hypothesizing, and modelling.

Suggested sequence of conceptual development activities

Unit	Topic	Focus
1	What is measurement?	Defining the concept of measurement
2	Attribute	Examining what attributes are and their presence across dimensions.
3	Unit	Developing a sense of the appropriateness of units and the need for unit standardization.
4	Number	Unpacking number as the product of measurement and how this quantifies attributes of geometric objects.
5	The fundamental principles of the measurement 'big idea'	Identifying fundamental principles that produce a conception of the measurement 'big idea'.

Conceptual Development Activities

Unit 1: What is measurement?

Content Standards: 1 – 6

Intent of this unit/activity

This activity serves to create a conception of what measurement is through defining measurement and applying this definition to the school mathematics context.

Habits of mind which are to be developed

The activity will speak to *invariance* and *quantification* as habits of mind.

Questions that can be asked

What is measurement?

Why do we measure?

What do we measure?

How do we measure?

How do we describe measurements?

What does the measurement/number represent?

How is measurement taught in Intermediate Phase classrooms?

Explanations

The engagement should be initiated through the question: *What is measurement?* This should allow the facilitator to gauge what the students' existing conceptions are of the content. By gauging their understandings, the facilitator would be able to adapt the presentation in relation their existing ideas or scaffold certain groups to afford all students similar foundations as a point of departure. In many cases, individuals find it difficult to field this question as they seldom would have thought about measurement in this way. Many would relate measurement to a skill or an activity rather than a concept and realizing that all valid perceptions should be affirmed. The focus would be to produce a generalizable definition or understanding of measurement that would afford individuals the ability to apply this definition to varying contexts, even those outside of the contexts familiar to them (e.g. capacity, length, area, etc.). In doing this, a conception of measurement starts to develop rather than the repertoire of unrelated skills or activities specific to a given contexts. Providing a definition as a reference would be a natural consequence of this question. The definition that supports the development of the materials in this resource is articulated as follows:

'*A measurement is a* ***number*** *that indicates a comparison between the* ***attribute*** *of an object being measured and the same attribute of a given* ***unit*** *of measure.*'

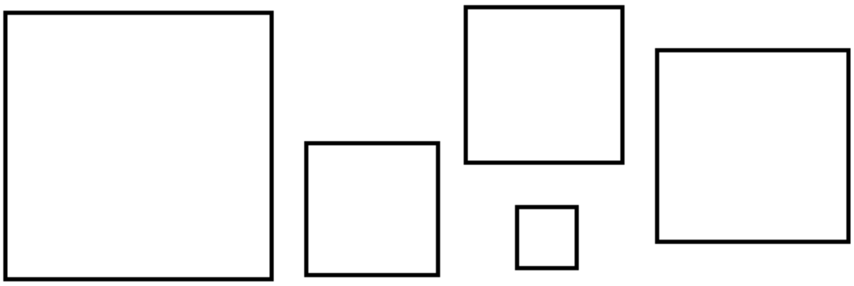
This definition is based on the definition developed by John Van de Walle as presented in the publication Elementary and Middle school Mathematics (2015).

Why do we measure? In fielding this question, the facilitator should focus on the properties of geometric elements (i.e. shapes and objects). Being able to distinguish one geometric element from another in term of dimension and their particular properties allows for categorization and generalization within categories. It is key to note that there is a logical sequence that relates to the teaching of geometry and measurement. The learners first need to have experienced the categorization of geometric elements as well as have a background in dimension before any meaningful work can be done in relation to measurement. This is because the language developed in describing geometric elements and dimension together with the concepts that embody this language will serve as the basis for conceptualizing measurement. The properties of geometric elements and the concept of dimension serves to **qualify** the objects that serve as the basis for study, whereas measuring these objects serves to **quantify** them. In this way, we are able to describe particular geometric object without any ambiguity around which specific object t we are referencing (in the absence of a diagram). Without measurement, we are able to describe a category of objects, yet would be unable to reference a specific object within the given category.

What do we measure? This question speaks to those geometric properties as isolated and described through engagements in classification and dimension. Measurement seeks to quantify particular elements within a given generalizable category, e.g. referencing a particular square within a group of squares, each possessing the properties that make it a square (i.e. quadrilateral, equiangular and equal-sided). Here a reference to size is what is required to make reference to a specific square within this category of squares.

Supporting and/or explanatory diagrams and videos

The quadrilaterals below are all squares



Describe each as if you are doing so to someone telephonically.
The intention is to enable the listener to draw these exact squares.
How would you do this?

The activity above teases out the need for measurement. The only way in which you would be able to provide accurate descriptions in order to re-create them would be if you measured each of the squares in order to distinguish one from the other. The key here is not to actually measure, but to describe how the problem could be solved. Of course, actually measuring the squares is a useful exercise but would negate the need for an explanation, which in this case, is essential.

The diagram above is used to tease out the importance of language and how language should be employed to direct the learner to particular aspects of the objects being measured (in the diagram this would be the comparison of the individuals' heights in the picture).

<u>Which materials/apparatuses are used?</u>

The use of the definition (as articulated previously) and the suggested diagrams/activity would suffice for this aspect of the content. Key to the engagement is getting the students thinking about what measurement is and how to teach it as a coherent concept across its varied applications (e.g. capacity, length, area, etc). The content should be facilitated in ways appropriate to the target audience, ensuring that they are able to relate to the contexts or references being employed. Facilitators are encouraged to use existing materials yet should pay close attention to the intention of this unit, i.e. defining what measurement is and justifying measure within the context of geometry, other than real-world applications or scientific measurement.

<u>Summary of unit</u>

Central to the activities and recommendations contained in this resource is presenting measurement as a concept. To enable this, we use a definition as a way of structuring the idea of what measurement is. Later Units will unpack key elements of this definition and apply them to contexts contained within school curricula.

Resources

You will need…

Research articles for support

References for further reading

Unit 2: Attribute

Content Standards: 1, 2, 4 and 5

Intent of this unit/activity

These activities attempt to make apparent the relation between the properties of geometric shapes and objects and the attributes of these shapes and objects employed in measurement. The activities also correlate specific attributes and their dimensionality, making explicit the differences between attributes in 1, 2 and 3 dimension and the language that supports these distinctions.

Habits of mind which are to be developed

These activities will speak to *invariance*, *quantification* and *visualization* as habits of mind.

Questions that can be asked

What is it that you want to measure?

What dimension is the property of the geometric figure that you are measuring in?

What attribute of the geometric figure are you measuring?

What is the extent of the attribute you are measuring?

Sequence and variety of questioning

Supporting and/or explanatory diagrams and videos

Explanations

The engagement needs to be premised by a recap of the properties of geometric shapes and objects. Emphasis needs to be put on specific language which distinguishes the dimensions in which these properties are perceived, given the fact that the same property could be afforded a different reference within an alternate dimension.

Dimension	Property reference
1-Dimension	Line
2-Dimension	Side
3-Dimesion	Edge

Students need to be cognizant of how the reference to the **'line'** (1-D) changes to **'side'** when working with a 2-D shape (square in the above table) and then further changes to **'edge'** within the context of a 3-D object (cube in the above table). This subtle, yet deliberate cues serve to make explicit the *dimensional context* within which the 'line' is employed. This reference becomes central to how the students isolate and perceive the attribute which would be the focus of a measurement activity.

The focus of these activities is to tease out all of the potential measurement attributes that constitute a particular everyday object, for which in this case, a bucket is used.

Q: What do you want to measure?

Being able to tease out with the students the potential attributes that could be measured would allow them insight into the ambiguity that could exist if the language a teacher employs does not make specific reference to the specific attribute being measured.

Q: How wide is the opening of the bucket?

In the above example, the question referencing the **1-D attribute** would be: "How wide is the opening of the bucket?" This measurement would reference a straight **line** from point to point.

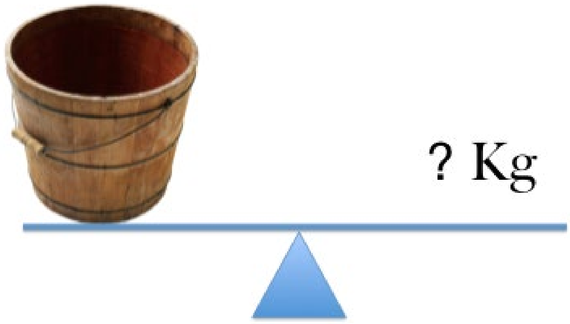
Q: What is the size of the outside surface of the bucket?

The measurement now references a **2-D attribute** of the same bucket. The word **'surface'** being referenced exists as a two-dimensional *plane* and makes explicit that a two-dimensional attribute of the bucket is being measured.

Q: How much liquid will it take to fill the bucket?

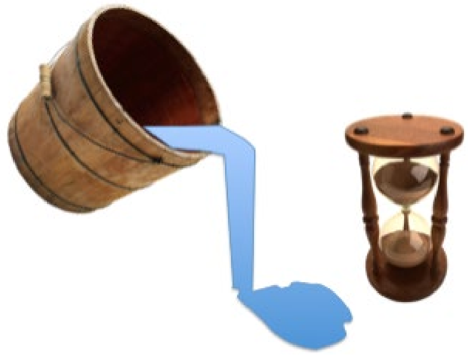
This measurement refers to a **3-D attribute** of the bucket. The question could also have been posed as: "What is the *volume* of the liquid inside the bucket?" Here, the word **'volume'** references a three-dimensional *space* and makes it explicit that a three-dimensional attribute of the bucket is being measured. Key to mention here is the issue that exists around the distinction between *volume* and *capacity*. Capacity normally references a receptacle and the potential volume that can be contained within it, e.g. the capacity of a jug (in liters) or the capacity of a stadium (in people). Volume speaks to the amount space a solid 3-D object occupies, normally measured in cubic millimeters, centimeters, meters, etc. (i.e. mm^3, cm^3, m^3).

Q: What is the mass of the bucket?



All three-dimensional objects are constituted of *matter* and as a result have **mass**. We distinguish between the mass of various **3-D objects** by observing the effect *gravity* has on them. This is referred to as **weight** and is measured in grams, milligrams, kilograms, etc. The distinction between mass and weight is often confused and inappropriately used in measurement contexts at school mathematics level.

Q: How much time would it take to empty the bucket?



Time is one of the most challenging measurement contexts to mediate due to its *abstract* nature. Witnessing time is impossible yet we are able to measure the *passing of time*. It is from this perspective that mediation needs to occur. Time as we measure it is entirely dependent on *celestial* movements, i.e. the movement of the planets and their satellites around the sun. We measure time in seconds, minutes, hours, days, weeks, months, etc. Part of the challenge in reading time is interpreting the *instruments* with which we measure time (i.e. clocks, wristwatches, cellphones, calendars, etc). Practicing the reading of **scales**, i.e. unitized incremental devices, is key in the teaching and learning of time and most other measurements.

<u>Which materials/apparatuses are used?</u>

At the start of the unit, having examples of two-dimensional shapes and three-dimensional objects would support the teaching engagement focused on introducing/recapping the properties of geometric figures. Using the same geometric figures would allow the students to apply the principles of dimensionality and afford opportunities to classify the properties of these geometric figures as existing in 1-, 2- and 3-Dimensions. It would be efficient to use a bucket in the discussion/activity involving the measurable attributes objects as it allows the facilitator to tease all of the attributes out of the single bucket. This would also ensure that the language employed in the engagement is specific and provides cues as to the specific attribute being attended to. This is deliberate and forms a central part of engagements in measurement, negating the ambiguity that often becomes a barrier to teaching measurement in school mathematics at this level. Examples of measuring instruments need to be at hand and referenced to ensure that the students are confident in using these teaching resources. (i.e. rulers, tape measures, scales, time keeping instruments, bottles, jugs, weights, etc).

How are these materials/apparatuses used?

Inside the classrooms at schools various measuring instruments need to be employed to familiarize learners with how these instruments work. Students we assume, are familiar with the use of these basic measuring instruments. For them it is far more important to focus on understanding measurement and how to employ these measurement contexts at schools to help develop their learners' conception of measurement.

Resources

You will need:

Rulers, tape measures, scales, time keeping instruments, bottles, jugs, weights, etc.

Research articles for support

References for further reading

Unit 3: Unit

Content standards: 1 – 6

Intent of this unit/activity

The focus of this unit is to develop an understanding around units of measurement. At this level, learners would need to develop deeper conceptions of measurement through discerning appropriate units of measure for different measurement contexts. Linking units of measure and the attributes of objects through dimensionality is a key part of this development, e.g. being able to see that a line is one-dimensional, thus requiring a one-dimensional unit of measure to quantify its length. Unitizing measurements serves as a context for learners to apply and extend their number knowledge and for them to perceive units in a practical way.

Habits of mind which are to be developed

These activities will speak to *invariance*, *quantification* and *visualization* as habits of mind.

Questions that can be asked

How would you go about measuring?

How can we compare the size of the attribute of one object to the size of the same attribute of another object?

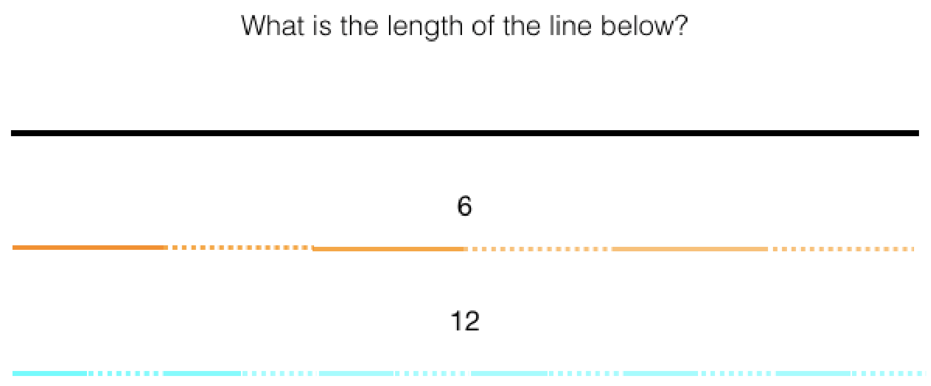
What would you use to measure the attribute of an object?

How could we go about describing the size of a 1-D, 2-D or 3-D geometric figure?

How can we tell if a given measurement is an appropriate way of quantifying/describing the size of the attribute being measured?

Explanations

In the Foundation Phase learners would have been exposed to the idea of unit standardization, or the need for a standard unit of measure to quantify/describe specific attributes of any given object. It may be appropriate for students to be taken through activities suitable for Foundation Phase learners that help develop the idea around the need for a standard unit. This will help the students gain insight into the types of problems that could arise out of the use of non-standard units and the ambiguity around these types of measurements. (e.g. getting the students to measure the length of the classroom using their foot lengths. This would result in varying measurements due to the varying shoe sizes. Possible questions could include: Which measurement is correct? Does the distance being measured change? What do the different measurements tell us? How could we avoid this type of measurement result?)



The above activity focusses on the same need to develop a standard unit. Using this activity, questions that follow could include:

Q: Which answer is correct?

Q: How do the sizes of the blue and orange units compare to each other?

Q: If you used a unit **three times the size** of the **blue units**, how would you describe the length of the line in relation to this new unit?

Q: Where does the first unit start?

Q: How do we arrange the units across the extent of the attribute being measured?

Q: What could create errors with the measurement being calculated?

The idea of this activity is to create an understanding between the **length of the units** and the **number of units** making up the length of the attribute. The students also need to understand that the length of the attribute being measured **does not change** and therefore remains constant. At a more general level, the way in which units are **iterated** across the attribute being measured should be a focus of the engagement. The fact that units cannot overlap or have unmeasured spaced between them should be stressed to ensure that **accuracy** is encouraged.

The idea of the **unit** you employ to measure needing to display the **same attribute** of the **object** being measured can also be extended across the different contexts or attributes previously discussed. What follows is an extension of the idea of using a single object (i.e. bucket) and correlating the unit of measure with the specific attribute being referenced.

To measure the attribute that references **distance** or a linear extent,
we use a we use a **1-dimensional** unit of measure

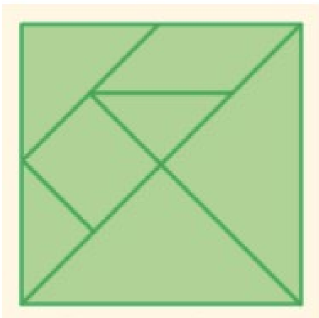
Q: What unit of measure would be appropriate to use in measuring the width of the opening of the bucket?

One dimensional length measurement requires units that measure distance. These units cover the particular length from point (origin) to point (termination). The focus of the engagement in this unit is to match units of measurement to the attribute being measured. The suitability of units could be discerned at two levels, i.e. in terms of dimension and also in terms of size (e.g. measuring the distance from Cape Town to Durban in millimeters would not be appropriate or using cm^2 to measure the volume of an object would not be appropriate).

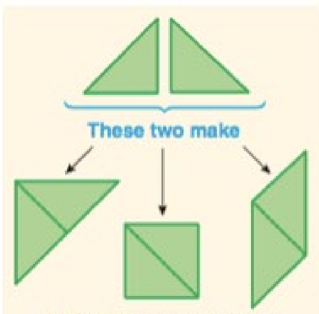
To measure the attribute that references the **area** of a plane or 2-dimensional extent, we need to use a **2-dimensional** unit of measure.

Q: What unit of measure would be appropriate to use in measuring the area covered by the outer surface of the bucket?

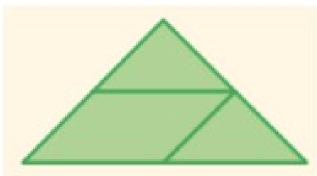
It is imperative that students are focused on conceptualizing what **area** is rather than clouding the concept of area with calculation strategies. Initially, learners in the classroom should be exposed to experiences that help them discern what **constitutes area** as a **dimensional** extent and in the context of an **attribute** of a geometric figure. Typical of these kinds of experiences could be captured in the activity that follows.



7 tangram shapes



The 2 small triangles make each of the medium shapes.



Van de Walle (et al), 2015

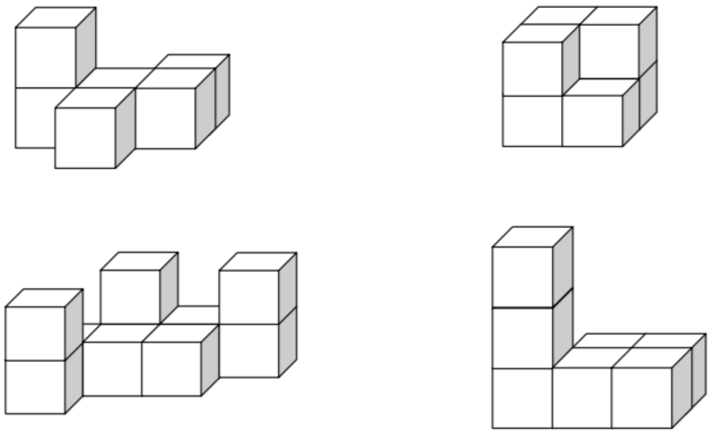
Q: How many of the small squares (as seen in the first diagram) would make up the area of the entire tangram?

To answer this question, students would need to reference the relationship described in the second diagram to measure the *area of the tangram* (i.e. **attribute**) using the *small triangles* as a **unit** of measure. They would need to consider that the area of the small square is made up of two of the small triangles and would conclude the **number** of squares making up the area of the entire tangram. Though this process, the focus of the students is specifically the **concept of area** or covering the entirety of the tangram, rather than using a formula to calculate the area of a square (which is what the tangram represents). This makes the numbers and calculations subordinate to the underlying principle, i.e. **what constitutes the area of the tangram?**

To measure the attribute that references the **volume** of an object or a 3-dimensional extent, we a need to use **3-dimensional** unit to measure

Q: What is the volume of the water contained in the bucket?

In order to answer this question, we need to use a unit that **occupies space** or possesses 3-D attributes to measure the **volume** of water in the bucket. Using stacking cubes and constructing block figures is an effective way of making the concepts of volume explicit without the need for calculation methods or arithmetic. Asking the learners to construct block figures like those below.



Q: Which of these block buildings has the largest volume?

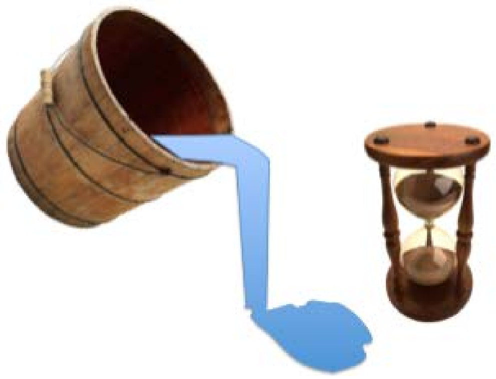
In answering this question, the *amount of space* each of these constructions occupy (i.e. **attribute**) is what needs to be compared. By identifying *how many* (i.e. **number**) cubes/blocks (i.e. **unit**) each construction consists of students can establish which construction has the largest volume. Key in this engagement is to get the students to appreciate that different looking arrangements may occupy the same volume of space. This would discourage the development any particular protype of what a particular amount of space looks like, allowing for a more robust and flexible interpretations and applications of volume.

Further extending this three-dimensional application, other qualities could require quantification, as seen in the extension of the attributes measured in the context of the bucket. Generally, attributes can be measured through quantifying either **intensive** or **extensive** quantities. At the level of the Intermediate Phase, learners initially have exposure only to extensive quantities.

Extensive quantities can be thought of as *quantities that change when the amount of measured substance changes*. This usually represents an **additive relation** in that two units of the same extensive quantities can be added, e.g. 5 m + 12 m = 17 m.

Intensive quantities *do not change by having more or less of the substance*. They normally cannot be measured directly and express a **multiplicative relationship** between two extensive quantities, e.g. density, speed and molecular weight.

To measure *time*, we need a unit that displays the same attribute.

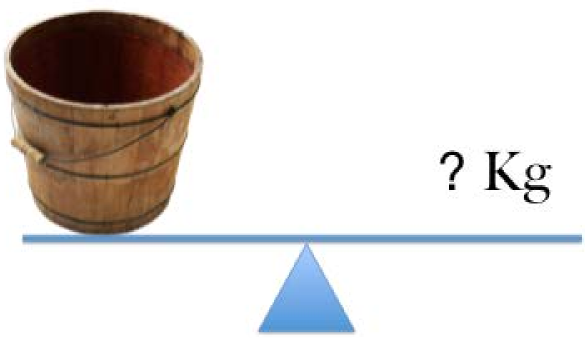


Q: How long would it take to empty the bucket?

If we assume that we could control the volume of water flowing from the bucket, we could measure how long (i.e. **number**) it would take for the water to be emptied (i.e. **attribute**) from the bucket. If measured in seconds (i.e. **unit**), then these would increase of decrease depending on the volume of water in the bucket, i.e. more water would result in more time taken. This principle would therefore classify **time** as an *extensive quantity*.

If we were to measure the **rate** of the flow of water from the bucket (i.e. **attribute**), it would be measured in milliliters per second (i.e. **unit**). This measurement would then be comprised of the multiplicative relation between volume and time, i.e. an *intensive quantity*.

To measure the total *mass*, we need a unit that displays the same attribute.



Q: What is the mass of the bucket?

The way in which we measure mass is by measuring the effect gravity has on mass. In the case of the bucket, we are attempting to measure the total mass of the substance from which the bucket is made (i.e. **attribute**). We therefore need to employ a unit that possesses mass or that is constituted of matter, e.g. kilogram (i.e. **unit**). So, the measurement will consist of the total amount of force (gravity) generated by the mass of the bucket (i.e. **number**) in kilograms.

<u>Which materials/apparatuses are used?</u>

The dominant context being employed in this unit is once again the bucket analogy, where the facilitator teases out various attributes relating to the physical properties of the bucket. Examples of measuring instruments need to be at hand and referenced to ensure that the students are familiar with the different types of units and confident in using these teaching resources. (i.e. rulers, tape measures, scales, time keeping instruments, bottles, jugs, weights, etc).

How are these materials/apparatuses used?

In this unit the intention is to promote that you could chose to measure any of the properties/attributes that the bucket possesses and to demonstrate that these attributes are inter-related. This would bring into focus the appropriateness of the unit being employed to quantify the extent of the attribute. Links between the dimensionality of the attribute and that of the unit also needs to be emphasized by the facilitator. Throughout the engagements in measurement, the definition employed earlier in this Teaching Unit is repeatedly emphasized through the repetition of the 3 key elements, i.e. **attribute**, **unit** and **number**. The purpose of this exposure is to ensure that every measurement context is framed by the definition for measurement so as to ensure that despite various contexts being employed, the same approach to measurement is adopted. This will promote coherence in measurement engagements and help develop a conception of what measurement is.

Resources

Bucket, rulers, tape measures, scales, time keeping instruments, bottles, jugs, weights, etc.

Research articles for support

References for further reading

Unit 4: Number

Content standards: 1 – 6

<u>Intent of this unit/activity</u>

This unit focuses on the number that is generated through the measurement process and what this number represents. In this unit the correlation between Number and Measurement as distinct aspects of the school mathematics curriculum becomes apparent. The key in facilitating this aspect of the Teaching Unit is to make evident how measurement can be perceived as a practical context for the development of number knowledge. The metric system is a decimal system which mimics our decimal number system. Teasing out the correlations between the two would help develop learners' number knowledge as well as affording them the opportunity to understand the measurements that are generated better. The ways in which the units are distinct notationally should also be emphasized as this is also based on arithmetic principles, i.e. cm, cm^2, cm^3. This speaks to the dimensionality of the attribute being measured as well as the unit being employed, making what has been measured apparent in the way in which the measurement is written. The use of scales is dominant throughout measurement engagements, modelling the use of a number line throughout measurement engagements. Reading scales is therefore essential in developing the number aspect of measurement engagements.

<u>Habits of mind which are to be developed</u>

These activities will speak to *invariance*, *quantification* and *visualization* as habits of mind.

<u>Questions that can be asked</u>

How do we describe a measurement?

What do the numbers in a measurement tell us about the attribute being measured?

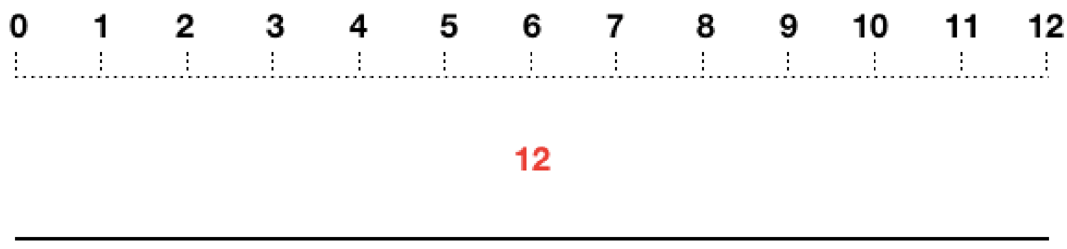
Do measurements only consist of whole numbers/units?

When we combine two measurements is it the same as adding in arithmetic?

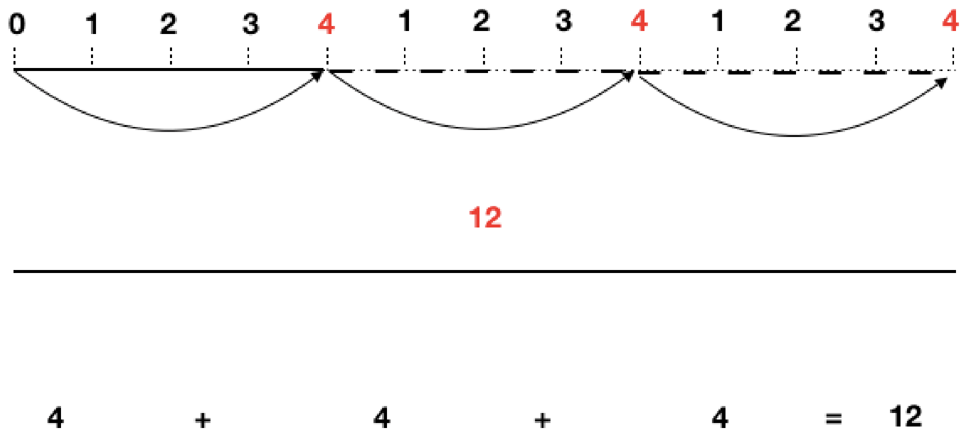
Do larger numbers always mean bigger measurements?

<u>Explanations</u>

It becomes evident that learners' number knowledge plays a central role in making sense of measurements. At the Foundation Phase level, learners were exposed to number within a measurement context through the use of a number line. The number line context formed a foundational resource for learners to employ when struggling with number or basic arithmetic. The number line itself represents a scale that is read from left to right. The units (ones from a number perspective) are stacked up against each other employing only the linear attribute of the resource. It becomes the perfect metaphor for measurement and sustains its number utility in its deployment within measurement.

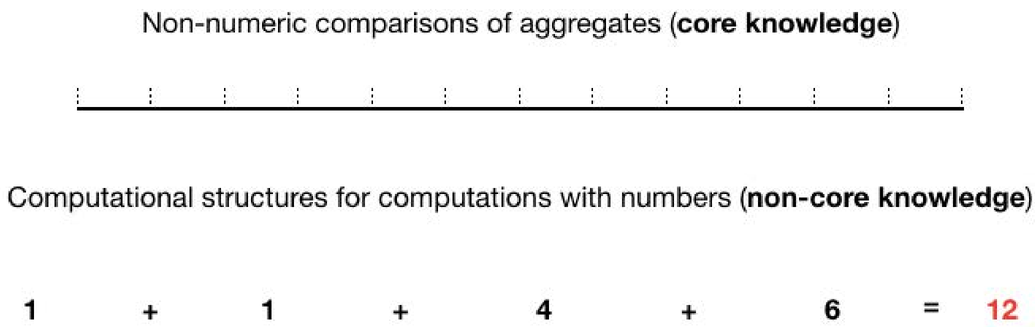


In the representation above, we can see the correlation between a line 12 units long and a number line that shows the ordinal numbers 1 to 12. As the number 12 is made up of all the preceding numbers, so learners should be able to conceptualize that from a line 12 units long, we could constitute a line 11 units long, or 10 units long, or 9 units long, etc. Being able to deconstruct this linear measurement in this way creates a robust conception of what a measurement represents.



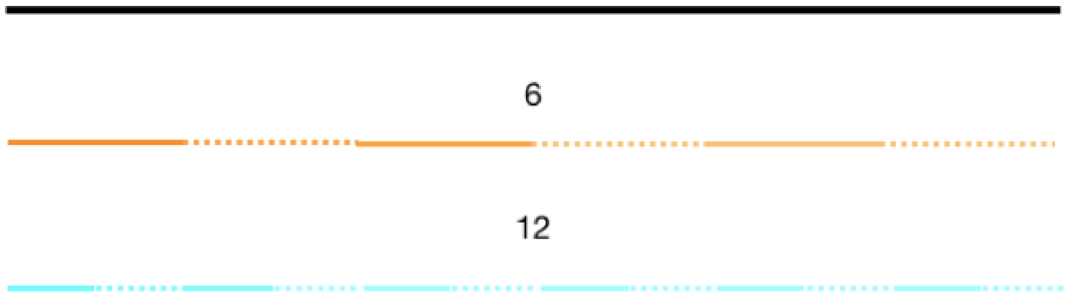
The diagram above seems to be reminiscent of Foundation Phase arithmetic where the teacher employs a number line on which to demonstrate the problem 4 + 4 + 4 = 12. In the Intermediate Phase Measurement context, the same resource could be employed, yet the rationale supporting the diagram would be: “For a line 12 units long, we can see that we could partition the line into three 4-unit lengths.” This promotes that learners perceive the number derived in a measurement as symbolizing a collection of same-sized individual units or smaller collections of these individual units. Whether the units employed are one-dimensional, two-dimensional or three-dimensional, the measurement attained represents a collection of identical units with the specific number of units representing the extent of the attribute being measured.

When learners directly compare the length of different objects, they are drawing on their core knowledge of non-numeric comparison of collections of units where a one-to-one comparison would enable them to discern which collection of aggregates is larger. This is the innate knowledge that every person is born with. It therefore makes sense for teacher to draw on this knowledge in order to develop the non-core knowledge involving computations with numbers.



It becomes clear that measurement can form a mediating function between these two types of knowledges. In the diagram below, the number line (scale) represents the core knowledge reference whereas the arithmetic below represents the non-core knowledge. Measurement will help the learners make sense of the arithmetic by employing the number line to compare the lengths of the units that make up a total of 12 units or even decompose 12 units into 1, 1, 4 and 6 units. Clearly, this type of correlation helps develop arithmetic as well as develop a better sense of units of measurement and the measurement process. Measurement and number are distinct processes but display correlations that allow for structure preservation.

Measurement does however create difficulties when specifying the extent of specific attributes using distinct units.



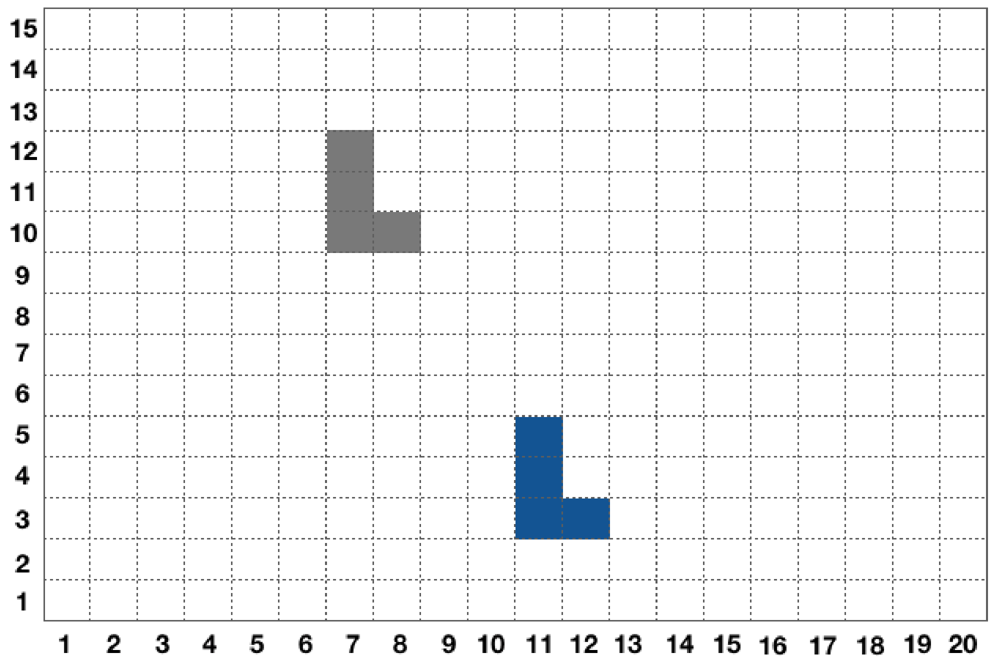
Q: Are the orange units and the blue units measuring the length of the same black line?

Q: If each of the measurements are referring to the length of the same line then does this mean that 6 = 12?

Q: What do these measurements tell us about the length of the orange unit and the length of the blue unit?

Learners should develop a robust sense of what measurements imply. In the example above, two distinct units are used to measure the length of the same black line. Learners should be able to interpret that because different measurements were attained for the length of the same line, it proves that different units were employed in each measurement. Further, they need to be able to discern that the size of the blue unit is half the size of the orange unit given the measurements that were derived. This kind of awareness would allow learners to become more fluent in conversions between various units, e.g. *mm* to *cm* to *m* to *km*. Again, because of the decimal basis of the metric system and knowledge of our decimal number place value system, learners should be afforded multiple opportunities to see the correlations between the two domains (i.e. measurement and number) in order to strengthen their knowledge within both these aspects of school mathematics.

In measurement, number can also indicate location within the Cartesian Plane context or as learners are introduced to in the primary school context, location on a grid.



The grid above represents the First Quadrant of the Cartesian Plane where both the horizontal rows (X-axis) and vertical columns (Y-axis) have positive **whole number references**. The numbers on each side of the grid originate in the bottom left corner and could be interpreted as a scale correlating each column or row with a particular number. The grey shape's location can then be described by the location of each of the grey squares making up the shape. In describing the top square of the grey shape, the reference or measurement **(7;12)** can be used. This specifies that the top square can be found where the **7th column** and the **12th row** meet. In effect, it provides a **measurement/number** whereby you can describe the exact location of the square. This measurement translates into: *'7 columns to the left of the point of origin and 12 rows up from the point of origin.'* Also, important to note is that in describing the co-ordinates (location) of the square, the horizontal/column reference is used first followed by the vertical reference/row. This is convention and in mathematics is referred to as a **co-ordinate** pair.

Q: Describe the location of the rest of the grey squares making up the grey shape using co-ordinate pairs.

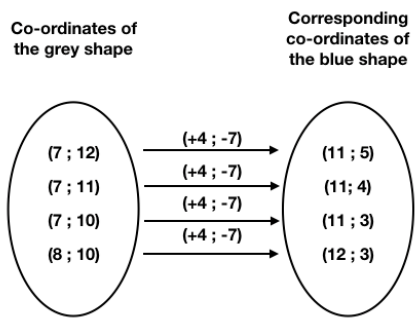
Q: What is the area of the grey shape?

Q: What unit did you use to measure the area of the grey shape?

Q: What is the perimeter of the grey shape?

Q: What unit did you use to measure the perimeter of the grey shape?

When we duplicate the shape in a different position, as represented by the blue shape, we can see that the location of the shape changes. In order for us to generate an identical copy, the change in co-ordinates need to be **consistent** for each of the squares making up the blue shape. This is demonstrated below.



From the mapping represented above, it becomes clear that the change in position from the grey shape to the blue shape can be described as **(+4 ; -7)**, or for each of the grey shape's co-ordinates, add 4 to the horizontal/column reference (measurement) and subtract 7 from the vertical/row reference (measurement). Through this comparison, we can see that a consistent change (the same across each of the co-ordinate pairs) in each of the co-ordinates/measurements describing each of the squares making up the grey shape results in the same shape constituted in an alternate position. In effect, a consistent change in the **numbers** (measurement) representing the location of each of the squares making up the grey shape results in an identical blue shape now found in a different position on the grid. An example of how **number** gives us key insights into the measurement of objects within particular environments/contexts.

<u>Which materials/apparatuses are used?</u>

Number lines and scales are primarily employed in this unit. The centrality of scales in most measurement contexts is articulated and it is promoted that learners are proficient in reading scales in various orientations and calibrations in order to measure accurately. Numbers, more specifically ordinality, is a resource that is deliberately used throughout the unit. For learners to acknowledge/identify the centrality of number in all measurements generated. This demonstrates how fundamental the relation is between number and measurement and allows students to conceptualize this relationship as reciprocal. Both linear (1-D) and area (2-D) measurement contexts are employed in the unit, yet this could be extended to all measurement contexts, provided that the focus should be on the number aspect of each of the measurements obtained.

<u>How are these materials/apparatuses used?</u>

In this unit, getting the students to interpret how number is central to any measurement is central to the engagement. The unit provides links to how measurement and number are interlinked and how meaningful and deliberate engagements in both aspects of school mathematics could potentially strengthen learners' number and measurement knowledge. The unit also clarifies what number knowledge learners' come in to the Intermediate Phase with and how teachers could potentially improve their conception of number as well as their conception of measurement through sustaining the links between these two conceptions.

Resources

Number lines, scales, grids, rulers, measuring tapes, etc.

Research articles for support

References for further reading

Unit 5: The fundamental principles of the measurement 'big idea'

Content standards: 1 – 6

<u>Intent of this unit/activity</u>

The intention of the activity is to tie together the preceding units and to present measurement as an interrelated concept in order to develop a coherent understanding of measurement to allow for consistent implementation across the various applications of measurement. This could be facilitated through the definition unpacked across the preceding units and the application of the 3 references central to this definition, i.e. **attribute**, **unit** and **number**. By deploying these aspects each instance measurement is engaged in, the teachers would be promoting the learners conceptualize measurement each time they deal with a different application of the content, e.g. length, area, volume, capacity, mass, time, etc. This unit also attempts to isolate generalizable principles that form the foundations for a robust understanding of what measurement is.

<u>Habits of mind which are to be developed</u>

These activities will speak to *invariance*, *quantification* and *visualization* as habits of mind.

<u>Questions that can be asked</u>

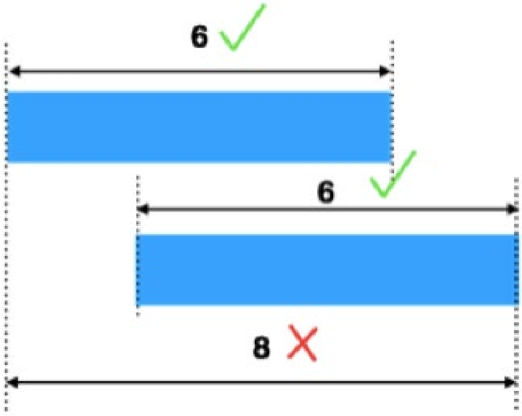
<u>Explanations</u>

The first of the generalizable principles relating to measurement could be considered to be **understanding the attribute**. Although this aspect is discussed in greater length in Unit 2, its significance needs to be established as central to understanding what measurement is. This principle **qualifies** exactly what it is you would be attempting to measure.

Q – If I wanted to measure the bucket, what about the bucket are you attempting to measure?

In the bucket application, it becomes apparent that the bucket possesses various measurable attributes. Through the use of appropriate language, we can articulate which attribute we are intending to measure. An appropriate question relating to measurement could be: 'What is the *size of the opening* of the bucket?' This question draws attention to exactly what attribute of the bucket is being measured. In the classroom, the learners should have no challenge in discerning what it is that needs to be measured, the learning should be around how the isolated attribute can be measured.

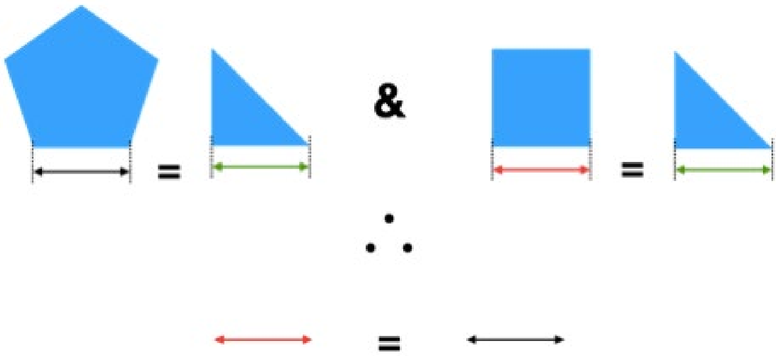
Another principle central to being able to conceptualize measurement is that of **conservation**. The learners' ability to conserve serves as an indicator to the extent to which they understand measurement and the properties of geometric figures. Conservation can be applied to measurement in 1-, 2- and 3-Dimension.



Q – Does the length of the blue bar change when the orientation of the bar changes?

In the example above, learners who are able to conserve would recognize that the length of the blue bar is constant (i.e. 6 units) despite the fact that the bar has been translated (i.e. moved forward) by 2 units. Learners who are unable to conserve would define the length of the transformed blue bar as 8 units. Learners who struggle to conserve would have difficulties with measurement and more than likely have conceptual gaps relating to the properties of geometric figures.

Transitivity is a general mathematical principle yet speaks directly to measurement within the varying contexts we employ in curriculum implementation. Transitivity entails identifying relations between (mathematical) objects and extending these relations to further applications or relations outside of those being directly compared.



Q – How do the lengths of the hexagon and the square compare?

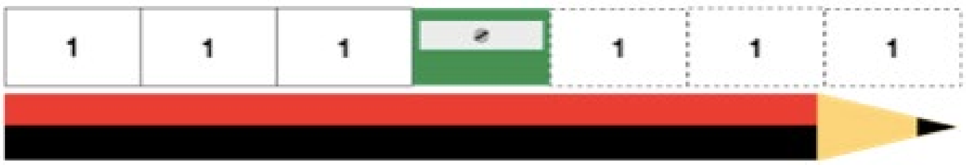
In the example above, the length of the side of the regular pentagon is equal to the length of the side of the equilateral triangle. It goes further to illustrate that the length of the side of the square equals the length of the side of the equilateral triangle. Because of this relationship, we can infer that the length of the side of the square equals the length of the side of the regular hexagon, despite the fact that these lengths had not been directly compared. This kind of extrapolation is a key principle in mathematics and plays a central role in measurement as in many cases the extents of various attributes are seldom directly compared. We use a measuring tool (e.g. tape measure, scale, etc.) to mediate quantifying these attributes and leaners need to be able to relate to these measurements both as *direct comparisons* and *indirect measurements*.

Equal partitioning is the next principle that is considered foundational in measurement. In linear measurement, the length/distance can be partitioned into different sized units.

Q – Is the length of each of the lines drawn above equal?

Learners need to identify that the length of the line being measured remains constant, yet the line could be broken up into identical units of varying lengths. It needs to become apparent that when employing smaller units, more of these units would cover the extent being measured as opposed to using larger units, resulting in less units covering the extent. In each case we need to re-enforce the fact that identical units need to be employed in each instance and that a combination of units can only be employed if there is a relationship established between the units, i.e. two of the smaller units are equal to a larger unit. Th possible units in each instance is limitless as this related to the number of fractions between two whole numbers.

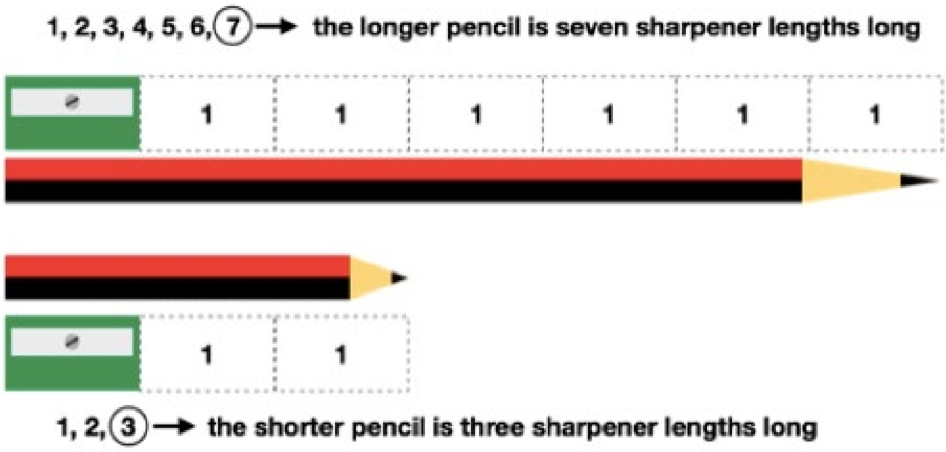
Units and iteration is the next principle to be promoted. This speaks to the fact that you could potentially use any unit/object which possesses the same attribute of the attribute being measured. The fact that this unit/object is repeatedly used (iterated) across the extent being measured makes for a credible measurement. The fact that there should be no spaces between the units/object, they should not overlap, and they should trace the path of the attribute being measured.



Q – How many pencil sharpener lengths does the length of the pencil measure?

Because the pencil sharpener and the pencil both possess length, we can use the length of the pencil sharpener to describe the length of the pencil. The measurement is established by iterating the length of the pencil sharpener along the length of the pencil and then counting the number of iterations to establish the measurement.

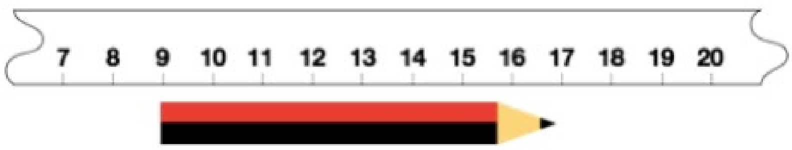
Accumulation of distance or **additivity** is a principle that applies to the cardinality of number as well as measurement contexts. It allows learners to apply the cardinality of numbers to this measurement context.



Q – How do the lengths of the two pencils above compare?

As in cardinality (as applied to sets and numbers), the learners need to realize that by counting each time the sharpener is iterated along the length of each pencil, the last sharpener length counted would be equal to the length of the respective pencil in sharpener lengths. This is a central idea in mathematics and generates the logic of the discipline. This serves as an appropriate context for learners to apply their knowledge of numbers, specifically cardinality or additivity.

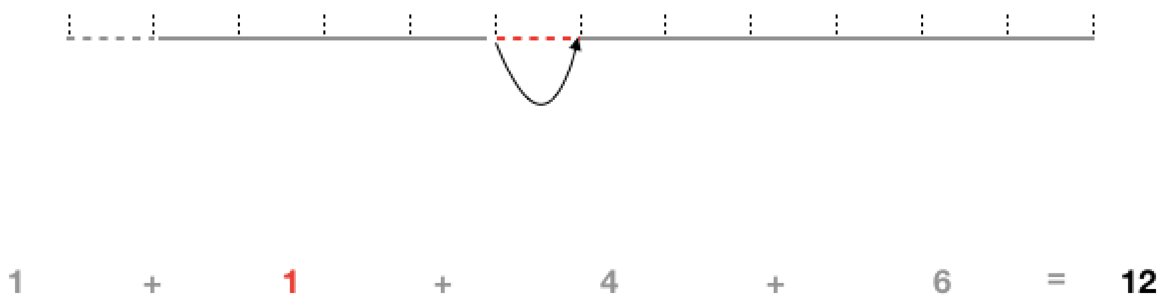
Origin is the next principle that needs careful attention. This refers to the point at which you start taking the measurement.



Q – What is the length of the pencil?

In the example above, the learners often define the length of the pencil as 17 units. Their focus seems to be entirely on the point at which the length of the pencil terminates, neglecting the origin which is clearly corelated to the 9th unit.

Finally, the **relation between number and measurement** is a principle that needs to be established to ensure that learners are competent at measurement. As alluded to throughout this document, the relationship between these two aspects of mathematics cannot be ignored and needs to be established to ensure a coherent understanding of both aspects and by extension mathematics. The one strengthens the other and difficulties experienced in either could result in barriers in both.



Q – What does the red unit in on the scale represent in the supporting equation?

For learners to be able to correlate the units of measure and values in an addition problem demonstrates their ability to model basic arithmetic and number knowledge. These two aspects of the mathematics curriculum support and strengthen each other in this way and allow for mathematics to be conceptualized as a coherent body of knowledge.

Which materials/apparatuses are used?

How are these materials/apparatuses used?

Resources

Research articles for support